Kernel-based performance evaluation of coded QAM systems

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RÉSUMÉ. Les estimateurs de taux d'erreur binaire par méthode à noyau sont d'un intérêt récent pour la réduction du coût des méthodes de Monte Carlo. Pour le moment, ils sont surtout appliqués à des modulations binaires. Dans ce papier, un estimateur à noyau est conçu pour des systèmes M-aires codés de Modulation d'Amplitude en Quadrature (MAQ). Les observations utilisées pour l'estimation sont définies sous forme de bits à valeurs souples bornées. Un noyau d'Epanechnikov est choisi et son paramètre de lissage obtenu sur la base du concept de bande passante canonique. Des simulations sont réalisées pour des systèmes MAQ-4 et MAQ-16 impliquant des canaux à bruit additif blanc Gaussien ainsi qu'à évanouïssements de Rayleigh. Les résultats obtenus montrent que l'estimateur proposé produit des gains en coût significatifs qui croissent avec E_b/N_0.

ABSTRACT. Kernel Bit Error Rate (BER) estimators are of recent interest for Monte Carlo sample size reduction. Until now, they mainly addressed binary modulation systems. In this paper, a kernel-based BER estimator is designed for coded M-ary Quadrature Amplitude Modulation (QAM) systems. The observations from which estimations are made are defined in the form of bounded soft bits. An Epanechnikov kernel function is selected and its smoothing parameter is derived based on the concept of canonical bandwidth. Simulations are run for 4-QAM and 16-QAM systems, involving additive white Gaussian noise and frequency-selective Rayleigh fading channels respectively. Simulation results show that the proposed estimator yields significative sample savings that grow with E_b/N_0.

MOTS-CLÉS : Taux d'erreur binaire, Estimateur à noyau, Méthode Monte Carlo, Fonction de densité de probabilité.

KEYWORDS : Bit Error Rate, Kernel estimator, Monte Carlo method, Probability density function.

1. Introduction

The Bit Error Rate (BER) is a measure of performance largely used in digital communications domain. Analytical BER estimation techniques have been studied [1], [2]. However, closed-form solutions are generally unavailable when considering complex digital communication systems. More successful have been simulation-based techniques at the core of which is the Monte Carlo (MC) method. The MC method is a universal technique that supplies an empirical determination of the BER estimate and that is commonly used as a reference for other methods. Its weak point is its high computational cost.

Since the 1970s, simulation-based techniques [3] were developed in order to reduce the sample size that the MC method requires to achieve accurate estimation. Recently, new BER estimation methods based on non-parametric *probability density function* (pdf) estimation have shown to achieve good performance for the uncoded binary-input Gaussian channel : namely Gaussian mixture models [4] and kernel estimators [5]. In [6], a kernel-based soft BER estimator is applied to Code Division Multiple Access (CDMA) schemes, for which efficient and reliable BER estimates have been reported. In [7], it is shown that kernel-based BER estimations can perform well in a blind way. Using Maximum Likelihood for the smoothing parameter optimisation, kernel method for BER estimation was applied to binary coded transmission schemes involving Turbo and Low Density Parity Check (LDPC) codes over CDMA systems [8].

To the best of our knowledge, BER estimation using kernel methods has been so far only applied to CDMA systems over Additive White Gaussian Noise (AWGN) channels. In this paper, we first address the issue of general M-ary modulations. Shifting from 2-ary real constellations to M-ary complex modulations involves the estimation of complex pdfs. As QAM systems are largely included in standards, we focus on this family of M-ary modulations. Secondly, we address the issue of estimating the BER when transmitting over frequency-selective fading channels. Hence, the distribution of the soft observations loses its Gaussian nature and finding an ad-hoc smoothing parameter for the kernel is not straightforward. In the remainder, we give a theoretical formulation of the Bit Error Probability (BEP) in Section 2 and present the principle of kernel-based estimation technique in Section 3. We describe the proposed kernel-based BER estimator in Section 4 while reporting simulation results in Section 5. In Section 6, we conclude the paper.

2. Theoretical formulation of the BEP

Let us consider a coded digital communication system that operates with Quadrature Amplitude Modulation (QAM) schemes. A signal containing coded QAM waveforms of alphabet $\{S_1, S_2, \dots, S_M\}$ is transmitted over a noisy channel. M is the constellation size. At the receiver-end, we assume that the channel decoder delivers N independent and identically distributed soft bits $(X_j)_{1\leq j\leq N}$. Let X denote the univariate real random variable that describes the soft bits $(X_j)_{1\leq j\leq N}$ and let $f_X^{(0)}$ (resp. $f_X^{(1)}$) be the conditional pdf of X such that the transmitted bit $b_i = 0$ (resp. $b_i = 1$). The BEP can be stated as :

$$
\begin{aligned}
p_e &= \Pr[X > 0, b_i = 0] + \Pr[X < 0, b_i = 1] && (1)\\
&= \Pr[X > 0 \,|\, b_i = 0]\Pr[b_i = 0] + \Pr[X < 0 \,|\, b_i = 1]\Pr[b_i = 1] && (2)\\
&= \pi_0 \int_0^{+\infty} f_X^{(0)}(x)\,\mathrm{d}x + \pi_1 \int_{-\infty}^{0} f_X^{(1)}(x)\,\mathrm{d}x, && (3)
\end{aligned}
$$

where π_0 and π_1 are the *a priori* probabilities of bits values "0" and "1" respectively.

The BER is an estimate of the BEP. Based on the MC approach, it is estimated by counting the errors that occured on the transmitted data. Based on the kernel technique, the principle of its estimation is described in the following.

3. Kernel-based soft BER estimation

In kernel-based BER estimation, the marginal conditional pdfs $f_X^{(0)}(x)$ and $f_X^{(1)}(x)$ are estimated as follows :

$$\hat{f}_X^{(b_i)}(x) = \frac{1}{n_{b_i}} \sum_{j=1}^{n_{b_i}} \frac{1}{h_{b_i}} K\left(\frac{x - X_j}{h_{b_i}}\right), \tag{4}$$

where K is any even regular pdf with zero mean and unit variance called the kernel, n_{b_i} is the cardinality of the subset of the soft observations $(X_j)_{1 \le j \le N}$ which are likely to be decoded into a binary "0" bit value (resp. "1") and h_{b_i} is a parameter called smoothing parameter (or bandwidth) that depends on the soft observations $(X_j)_{1 \le j \le n_{b_i}}$. Then, p_e in Eq. (3) can be estimated as

$$\hat{p}_e = \pi_0 \int_0^{+\infty} \hat{f}_X^{(0)}(x)\,\mathrm{d}x + \pi_1 \int_{-\infty}^{0} \hat{f}_X^{(1)}(x)\,\mathrm{d}x. \tag{5}$$

The choice of the kernel K is related to the density function under estimation. Whenever the observed samples are distributed over a large scale, distributions with an infinite support (e.g., Gaussian distribution) are well suited. However, finite support distributions such as Epanechnikov or Quartic distributions should be selected to model K when the observed samples are bounded.

The design of the smoothing parameter h is a major issue since it significantly governs the accuracy of the estimation. To the end, optimisation of h with respect to some given constraints has been proposed. One of the most popular is the Asymptotic Mean Integrated Squared Error (AMISE) criterion. When the AMISE criterion is used, the optimal smoothing parameter is derived [9] as,

$$h^*_{\text{AMISE}} = \left[\frac{\int K^2(x)\,\mathrm{d}x}{\int f_X''(x)^2\,\mathrm{d}x \left(\int x^2 K(x)\,\mathrm{d}x\right)^2}\right]^{1/5} N^{-1/5}, \tag{6}$$

where $f_X''(x)$ is the second derivative of the pdf $f_X(x)$. Clearly, the constraint in Eq. (6) is the prior knowledge of the target distribution f_X, which is of course unknown and searched for. In practice, some reference distribution can be used to replace f_X, with mean and variance matching those of data. In the literature, the Gaussian distribution is a popular choice for f_X. Many designs of h^*_{AMISE} can be found including this recent one given as follows [10] :

$$h^*_{\text{Gau}} = (4/3)^{1/5} \min\left(\hat{\sigma}, IQR/1.34\right) N^{-1/5}. \tag{7}$$

where $\hat{\sigma}$ is the standard deviation of the data and IQR is their interquartile range.

4. Proposed kernel-based BER estimator scheme

Let us consider a digital communication system that includes a channel codec (encoder/decoder). The coded BER is the BER that is determined at the output of the channel decoder. A kernel-based soft coded BER estimator is proposed in this paper. Suited soft bits have to be given at the entry of the estimator. We define the soft bits as follows :

$$X_j = \Pr[b_j = 1|r] - \Pr[b_j = 0|r], \tag{8}$$

where r is the received signal. Let us assume that the channel decoder requires soft inputs in the form of Log-Likelihood Ratio (LLR). Each M-ary QAM soft symbol at the output of the channel carries $k = log_2(M)$ LLR bits $(L_j)_{1\leq j\leq k}$ that can be retrieved by a symbol-to-bit soft demapping [11]. We also assume that the outputs of the channel decoder are soft LLR bits. The *j*th LLR, L_j, is defined as

$$L_j = \log\left(\frac{\Pr[b_j = 1|r]}{\Pr[b_j = 0|r]}\right). \tag{9}$$

From Eq. (8), Eq. (9) and constraint $\Pr[b_j = 1|r] + \Pr[b_j = 0|r] = 1$, the soft bit X_j is derived in terms of the channel decoder output L_j as follows :

$$X_j = \frac{1 - e^{-L_j}}{1 + e^{-L_j}}. \tag{10}$$

Using the soft bits $(X_j)_{1\leq j\leq N}$, the proposed kernel-based estimator can perform, provided a kernel function K and a suitable smoothing parameter h are selected.

As shown in Eq. (10), the soft bits $(X_j)_{1\leq j\leq N}$ are bounded between -1 and $+1$. So, among the kernel function with bounded support, the Epanechnikov kernel function $K(x) = \frac{3}{4}\left(1 - x^2\right) I(|x| \leq 1)$ is chosen. Then it can be checked that the kernel estimator with bandwith h will be restricted to interval $[-1-h, 1+h]$. Since optimaly chosen h remains much smaller than 1 for large samples, we can consider that numericaly the support constraint for the distribution of X is satisfied when using the Epanechnikov kernel. Therefore, we need to find the corresponding smoothing parameter h^*_{Epa} that approximates well h^*_{AMISE} of Eq. (6). As h^*_{Gau} is a good approximation of h^*_{AMISE} of Eq. (6) in the context of a Gaussian kernel, the idea is to derive h^*_{Epa} from h^*_{Gau} based on the concept of canonical bandwidth [12]. The parameter h^*_{Epa} is then expressed as

$$h^*_{\text{Epa}} = \frac{\delta_{Epa}}{\delta_{Gau}} h^*_{\text{Gau}}, \tag{11}$$

where, from [12] $\delta_{Gau} \approx (1/4)^{1/10} = 0.7764$ and $\delta_{Epa} \approx 15^{1/5} = 1.7188$ are the canonical bandwidths of the Gaussian and Epanechnikov kernels.

At this stage, the expressions of the two marginal conditional pdfs $\hat{f}_X^{(0)}(x)$ and $\hat{f}_X^{(1)}(x)$ can be derived from Eq. (4) and then, Eq. (5) can be rewritten as follows :

$$\hat{p}_e = \pi_0 \int_0^{+\infty} \frac{1}{n_0} \sum_{j=1}^{n_0} \frac{1}{h_0^*} K\left(\frac{x - X_j}{h_0^*}\right) \mathrm{d}x + \pi_1 \int_{-\infty}^{0} \frac{1}{n_1} \sum_{j=1}^{n_1} \frac{1}{h_1^*} K\left(\frac{x - X_j}{h_1^*}\right) \mathrm{d}x, \tag{12}$$

where h_0^* (resp. h_1^*), computed according to Eq. (11), is the selected optimal bandwidth which will govern the estimation accuracy of $\hat{f}_X^{(0)}(x)$ (resp. $\hat{f}_X^{(1)}(x)$). After transformations that are detailed in Appendix, Eq. (12) leads to the expression of the coded BER estimate as follows :

$$\hat{p}_e = \frac{\pi_0 L_0}{n_0} + \frac{\pi_1 L_1}{n_1} + \sum_{\substack{|\alpha_j|\leq 1,\\ 1\leq j\leq n_0}} \frac{3\pi_0}{4n_0}\left(\frac{2}{3} - \alpha_j + \frac{\alpha_j^3}{3}\right) + \sum_{\substack{|\beta_j|\leq 1,\\ 1\leq j\leq n_1}} \frac{3\pi_1}{4n_1}\left(\frac{2}{3} + \beta_j - \frac{\beta_j^3}{3}\right), \tag{13}$$

where $\alpha_j = -X_j/h_0^*$, $\beta_j = -X_j/h_1^*$, L_0 (resp. L_1) is the cardinality of the subset of $(\alpha_j)_{1\leq j\leq n_0}$ (resp. $(\beta_j)_{1\leq j\leq n_1}$) which are less than -1 (resp. greater than 1). Based on Eq. (13), coded BER estimates can be evaluated using soft bits $(X_j)_{1\leq j\leq N}$.

5. Simulation results

The proposed estimator has been simulated on a single-carrier QAM transmission scheme over the AWGN channel and also on a multi-carrier QAM transmission scheme over a frequency-selective Rayleigh fading channel. A Gray-coded 4-QAM and 16-QAM constellations were considered. The Rayleigh channel was ten taps long with a sample period of $12.8\mu s$, an $8Hz$ maximum Doppler shift and average taps gains given in $watts$ by the vector $[0.0616\ 0.4813\ 0.1511\ 0.0320\ 0.1323\ 0.0205\ 0.0079\ 0.0778\ 0.0166\ 0.0188]$. To mitigate inter-symbol and inter-carrier interferences, a Cyclic Prefix (CP) Orthogonal Frequency Division Multiplexing (OFDM) technique was implemented. The length of the CP was set to 9 and the number of OFDM sub-carriers set to 128. A 128-point FFT (Fast Fourier Transform) was performed. The Channel codec was a $4/7$-rate LDPC code with a Gallager-based parity check matrix built to be of rank 15. The number of iterations was set to 10 (resp. 30) for the AWGN (resp. Rayleigh) channel. An Epanechnikov kernel function and the smoothing parameter of Eq. (11) were selected.

We evaluate the performance in terms of absolute bias and Confidence Interval (CI). The absolute bias is defined as $|\,\mathbb{E}[\hat{p}_e] - p_e\,|$ where $\hat{p}_e$ represents an estimate of the coded BER. The true BER p_e is computed in the form of a benchmark using MC simulations. The CI has been calculated for a 95% confidence level. To validate the proposed estimator over the AWGN channel, *Figure* 1 offers a visual way to evaluate the bias for 4-QAM and 16-QAM transmission schemes. We can see that the kernel-based coded BER estimates data points are very close to the true BER (benchmark) from values greater than 10^{-1} down to 10^{-5}. *Table* 1 illustrates the bias and the CI using numerical data related to 4-QAM system simulation. From the observed CIs and their corresponding kernel sample sizes N_K, we derived (see [3]) the required sample sizes for MC simulations to yield equal performance and noted sample savings up to a factor 16. As for the performance achieved over the Rayleigh channel, the green curves with diamond marks in *Figure* 2 illustrate that coded BER estimates are close to their corresponding benchmarks. Detailed information about the bias, the CIs and the sample sizes is provided in *Table* 2 as far as 16-QAM transmission schemes are concerned. A thorough analysis of the observed numerical data let us notice that all the data points on the green curves are associated to coded BER values that fall into their corresponding CIs. The observed smallest CI is $[0.89p_e,\ 1.11p_e]$ and the largest of all is $[0.52p_e,\ 1.48p_e]$. If we considered $[0.50p_e,\ 1.50p_e]$ as the largest CI over which the estimator is declared not reliable and combining with the fact that all the mean values of the BER estimates are inside their corresponding CIs, we can conclude, at the light of the observed CIs, that the proposed estimator is reliable for BER values down to the neighbourhood of 10^{-4}.

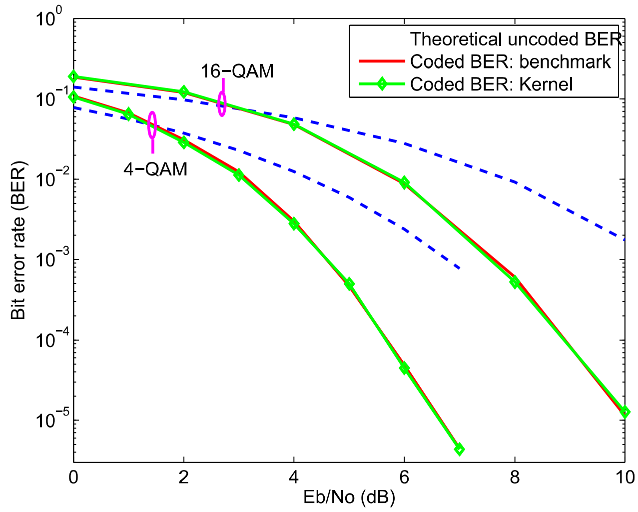

Figure 1. *Performance of the proposed estimator over the AWGN channel*

Regarding the efficiency, the two last Columns of *Table* 2 show that the proposed estimator requires less samples than the MC method. The given kernel (N_k) and MC (N_{mc}) sample sizes are those required for the two methods to achieve (almost) equal bias and CI. To illustrate this, let us consider the row of $E_b/N_0 = 12\,dB$ in *Table* 2. The proposed estimator achieved an efficiency described by a sample size of $50\,000$ against $127\,995$ for the MC estimator. In the same time, the proposed estimator achieved a CI of $[0.81p_e,\ 1.19p_e]$ versus $[0.80p_e,\ 1.20p_e]$ for the MC estimator. The two estimators performed the estimation with almost equal bias (0.0011 for the MC method against 0.0012 for the proposed kernel method). Moreover, for $E_b/N_0 = 20\,dB$ in *Table* 2, both the MC and the proposed estimators performed an estimate with equal bias and achieved CIs are $[0.62p_e,\ 1.38p_e]$ for the MC estimator against $[0.67p_e,\ 1.33p_e]$ for the proposed one. The corresponding sample saving achieved by the proposed estimator is at least of a factor 5.

Behind this efficiency of the proposed estimator is also hidden its performance in terms of the power consumption. The MC method and the proposed estimator yield almost equal CPU time for equal sample sizes ; e.g. : at $E_b/N_0 = 20\,dB$ and for a sample size of

Tableau 1. *Numerical results of coded 4-QAM BER estimation over AWGN channel*

E_b/N_0	Benchmark	Bias	CI	N_K
$00\,dB$	1.1×10^{-1}	0.03×10^{-1}	$[0.94p_e,\ 1.06p_e]$	10^3
$01\,dB$	6.7×10^{-2}	0.22×10^{-2}	$[0.90p_e,\ 1.10p_e]$	10^3
$02\,dB$	3.1×10^{-2}	0.22×10^{-2}	$[0.82p_e,\ 1.18p_e]$	10^3
$03\,dB$	1.2×10^{-2}	0.11×10^{-2}	$[0.93p_e,\ 1.07p_e]$	10^4
$04\,dB$	3.0×10^{-3}	0.18×10^{-3}	$[0.81p_e,\ 1.19p_e]$	10^4
$05\,dB$	4.7×10^{-4}	0.30×10^{-4}	$[0.89p_e,\ 1.11p_e]$	10^5
$06\,dB$	4.9×10^{-5}	0.38×10^{-5}	$[0.66p_e,\ 1.34p_e]$	10^5
$07\,dB$	4.4×10^{-6}	0.09×10^{-6}	$[0.54p_e,\ 1.46p_e]$	10^6

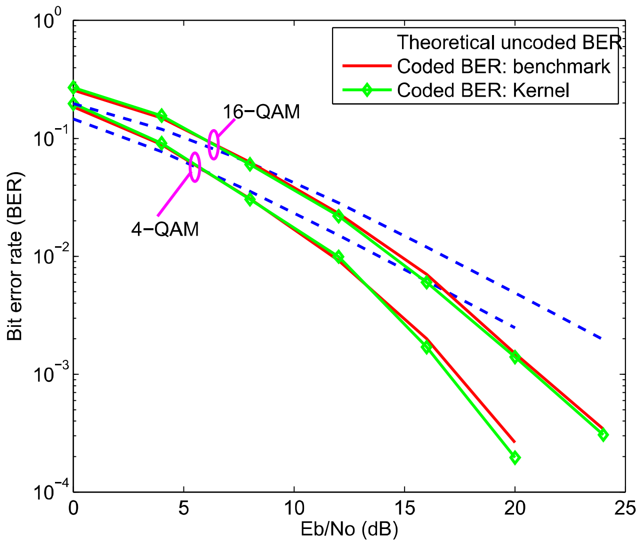

Figure 2. *Performance of the proposed estimator over Rayleigh channel*

$100\,000$, the CPU time engendered over the Rayleigh channel is 33.24 seconds for the MC method against 35.27 seconds for the proposed estimator. However, when the sample size increases it causes the CPU time to increase too. So, the sample saving due to the kernel method is beneficial in terms of power consumption. As an illustration, the performance achieved at $E_b/N_0 = 24\,dB$ (see Table 2) is at the cost of a CPU time of 7.27 minutes for the proposed estimator while being by far greater than 4.35 hours for the MC method.

6. Conclusion

In this paper, we proposed a kernel-based coded bit error rate estimator involving soft M-ary Quadrature Amplitude Modulation (QAM) symbols. An Epanechnikov kernel function was selected. The corresponding smoothing parameter was determined based on the concept of canonical bandwidth. Simulation results were reported for coded 4-QAM and 16-QAM single carrier transmissions over the additive white Gaussian noise channels and for coded multiple carrier modulations over a frequency-selective Rayleigh fading channel. Through curves and numerical data, the proposed kernel-based estimator showed to be, for equal reliability, more efficient than the Monte Carlo estimator. In future works, we will be interested in the possible efficiency improvement that might be achieved if different bandwidth selection strategies were implemented.

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Tableau 2. *Numerical results of coded* 16*-QAM BER estimation over Rayleigh channel*

E_b/N_0	Benchmark	Bias	CI	N_k	N_{mc}
$00\,dB$	2.58×10^{-1}	0.13×10^{-1}	$[0.89p_e,\, 1.11p_e]$	1.0×10^3	3.0×10^3
$04\,dB$	1.50×10^{-1}	0.06×10^{-1}	$[0.86p_e,\, 1.14p_e]$	2.0×10^4	1.9×10^4
$08\,dB$	6.28×10^{-2}	0.26×10^{-2}	$[0.87p_e,\, 1.13p_e]$	5.0×10^4	5.1×10^4
$12\,dB$	2.31×10^{-2}	0.12×10^{-2}	$[0.81p_e,\, 1.19p_e]$	5.0×10^4	1.3×10^5
$16\,dB$	7.00×10^{-3}	1.00×10^{-3}	$[0.73p_e,\, 1.27p_e]$	5.0×10^4	1.0×10^5
$20\,dB$	1.50×10^{-3}	0.08×10^{-3}	$[0.67p_e,\, 1.33p_e]$	1.0×10^5	$> 5.1 \times 10^5$
$24\,dB$	3.42×10^{-4}	0.36×10^{-4}	$[0.54p_e,\, 1.46p_e]$	4.1×10^5	$> 2.6 \times 10^6$

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8. Appendix

The BER estimate as given in Eq. (12) is

$$\hat{p}_e = \pi_0 \int_0^{+\infty} \frac{1}{n_0} \sum_{j=1}^{n_0} \frac{1}{h_0^*} K\left(\frac{x - X_j}{h_0^*}\right) \mathrm{d}x + \pi_1 \int_{-\infty}^{0} \frac{1}{n_1} \sum_{j=1}^{n_1} \frac{1}{h_1^*} K\left(\frac{x - X_j}{h_1^*}\right) \mathrm{d}x, \tag{A.1}$$

where n_0 (resp. n_1) is the cardinality of the subset of the soft observations among $(X_j)_{1\leq j\leq N}$ which are likely to be decoded into a binary "0" bit value (resp. "1") and h_0^* (resp. h_1^*) is the selected optimal smoothing parameter which will govern the accuracy of the estimation of $\hat{f}_X^{(0)}(x)$ (resp. $\hat{f}_X^{(1)}(x)$). More explicitly, as $K(x) = \frac{3}{4}\left(1 - x^2\right) I(|x| \leq 1)$, we have

$$\begin{aligned} \hat{p}_e = & \tfrac{\pi_0}{n_0} \int_0^{+\infty} \sum_{j=1}^{n_0} \tfrac{3}{4h_0^*} \left[1 - \left(\tfrac{x-X_j}{h_0^*}\right)^2\right] I\left(\left|\tfrac{x-X_j}{h_0^*}\right| \leq 1\right) \mathrm{d}x \\ & + \tfrac{\pi_1}{n_1} \int_{-\infty}^{0} \sum_{j=1}^{n_1} \tfrac{3}{4h_1^*} \left[1 - \left(\tfrac{x-X_j}{h_1^*}\right)^2\right] I\left(\left|\tfrac{x-X_j}{h_1^*}\right| \leq 1\right) \mathrm{d}x. \end{aligned} \tag{A.2}$$

Then, using one of the properties of the integral, we get

$$\begin{aligned} \hat{p}_e = & \tfrac{\pi_0}{n_0} \sum_{j=1}^{n_0} \int_0^{+\infty} \tfrac{3}{4h_0^*} \left[1 - \left(\tfrac{x-X_j}{h_0^*}\right)^2\right] I\left(\left|\tfrac{x-X_j}{h_0^*}\right| \leq 1\right) \mathrm{d}x \\ & + \tfrac{\pi_1}{n_1} \sum_{j=1}^{n_1} \int_{-\infty}^{0} \tfrac{3}{4h_1^*} \left[1 - \left(\tfrac{x-X_j}{h_1^*}\right)^2\right] I\left(\left|\tfrac{x-X_j}{h_1^*}\right| \leq 1\right) \mathrm{d}x. \end{aligned} \tag{A.3}$$

Now, let us set the following changes of variables :

$$\begin{cases} u = \frac{x-X_j}{h_0^*} \\ v = \frac{x-X_j}{h_1^*}. \end{cases}$$

We obtain

$$\begin{aligned} \hat{p}_e = & \tfrac{3\pi_0}{4n_0} \sum_{j=1}^{n_0} \int_{-X_j/h_0^*}^{+\infty} \left(1 - u^2\right) I\left(|u| \leq 1\right) \mathrm{d}u \\ & + \tfrac{3\pi_1}{4n_1} \sum_{j=1}^{n_1} \int_{-\infty}^{-X_j/h_1^*} \left(1 - v^2\right) I\left(|v| \leq 1\right) \mathrm{d}v, \end{aligned} \tag{A.4}$$

and then,

$$\hat{p}_e = \frac{3\pi_0}{4n_0} \sum_{j=1}^{n_0} \int_{[\alpha_j,\, +\infty] \cap [-1,\, 1]} \left(1 - u^2\right) \mathrm{d}u + \frac{3\pi_1}{4n_1} \sum_{j=1}^{n_1} \int_{[-\infty,\, \beta_j] \cap [-1,\, 1]} \left(1 - v^2\right) \mathrm{d}v, \tag{A.5}$$

where $\alpha_j = -X_j/h_0^*$ and $\beta_j = -X_j/h_1^*$. Depending on the values of α_j (resp. β_j), three cases are possible among which one leads to zero ; hence we get,

$$\hat{p}_e = \frac{3\pi_0}{4n_0}\left\{\sum_{\substack{\alpha_j<-1,\\ 1\le j\le n_0}}\left[t-\frac{t^3}{3}\right]_{-1}^{1}+\sum_{\substack{|\alpha_j|\le 1,\\ 1\le j\le n_0}}\left[t-\frac{t^3}{3}\right]_{\alpha_j}^{1}\right\}$$
$$+\frac{3\pi_1}{4n_1}\left\{\sum_{\substack{\beta_j>1,\\ 1\le j\le n_1}}\left[t-\frac{t^3}{3}\right]_{-1}^{1}+\sum_{\substack{|\beta_j|\le 1,\\ 1\le j\le n_1}}\left[t-\frac{t^3}{3}\right]_{-1}^{\beta_j}\right\}. \tag{A.6}$$

Finally, the BER estimate expression is as follows :

$$\hat{p}_e = \frac{\pi_0 L_0}{n_0}+\frac{\pi_1 L_1}{n_1}+\frac{3\pi_0}{4n_0}\left\{\sum_{\substack{|\alpha_j|\le 1,\\ 1\le j\le n_0}}\left(\frac{2}{3}-\alpha_j+\frac{\alpha_j^3}{3}\right)\right\}$$
$$+\frac{3\pi_1}{4n_1}\left\{\sum_{\substack{|\beta_j|\le 1,\\ 1\le j\le n_1}}\left(\frac{2}{3}+\beta_j-\frac{\beta_j^3}{3}\right)\right\}, \tag{A.7}$$

where L_0 (resp. L_1) is the cardinality of the subset of $(\alpha_j)_{1\le j\le n_0}$ (resp. $(\beta_j)_{1\le j\le n_1}$) which are less than -1 (resp. greater than 1).