Global weak solution to a 3-D Kazhikhov-Smagulov model with Korteweg stress

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ABSTRACT. In this article, we consider a multiphasic incompressible fluid model, called the Kazhikhov-Smagulov model, with a specific stress tensor which depends on density derivatives, introduced by Korteweg. We establish the existence of global weak solution to this model in a 3D bounded domain.

RÉSUMÉ. Dans cet article, nous considérons un modèle de fluide incompressible multiphasique, appelé modèle de Kazhikhov-Smagulov, avec un tenseur de contraintes spécifique qui dépend des dérivées d'ordre élevé de la densité, introduit par Korteweg. Nous établissons l'existence d'une solution faible globale pour ce modèle dans un domaine borné en 3D.

KEYWORDS : Kazhikhov-Smagulov model, Korteweg model, weak solution, global existence result.

MOTS-CLÉS : Modèle de Kazhikhov-Smagulov, modèle de Korteweg, solution faible, existence globale.

1. Introduction

We are concerned with systems of PDEs describing the evolution of mixture flows. Let Ω be a bounded open set in $\mathbb{R}^3$ with boundary Γ that is regular enough and let $\boldsymbol{n}$ be the outwards unit normal on the boundary Γ. We denote by $[0,T]$ the time interval, for $T>0$. The mixture of two fluids is described by the density $\rho(t,\boldsymbol{x})\geq 0$, the velocity field $\boldsymbol{v}(t,\boldsymbol{x})\in\mathbb{R}^3$ and the pressure $p(t,\boldsymbol{x})$, depending on the time and space variables $(t,\boldsymbol{x})\in[0,T]\times\Omega$. According to [4, 7, 8], we consider the Korteweg equations for generalized incompressible fluids whose density and volume change with the concentration $\phi(t,\boldsymbol{x})\geq 0$ and eventually the temperature, but not with pressure. In general, the velocity field $\boldsymbol{v}$ of such incompressible fluids is not solenoidal, $\operatorname{div}\boldsymbol{v}\neq 0$. Assuming that each fluid is incompressible, the mass density is conserved in the absence of diffusion. The theory of Korteweg, introduced in [9], considers the possibility that stresses are induced by gradients of concentration and density in a slow process of diffusion of incompressible miscible liquids. Such stresses could be important in regions of high gradients and they mimic the surface tension.

In order to model the fluid capillarity effects, Korteweg introduced in the usual compressible fluid model a specific stress tensor which depends on density derivatives. Following the rigorous formulation presented in [4] (see also [2]) and neglecting thermal fluctuations, the model reads

$$\left\{\begin{array}{l}\partial_t\rho+\operatorname{div}(\rho\boldsymbol{v})=0,\\ \partial_t(\rho\boldsymbol{v})+\operatorname{div}(\rho\boldsymbol{v}\otimes\boldsymbol{v})=\rho\boldsymbol{g}+\operatorname{div}(\boldsymbol{S}+\boldsymbol{K}),\end{array}\right. \tag{1}$$

where $\boldsymbol{g}$ stands for the gravity acceleration (but it can include further external forces). The viscous stress tensor $\boldsymbol{S}$ and the Korteweg stress tensor $\boldsymbol{K}$ are given by :

$$\left\{\begin{array}{rcl}\boldsymbol{S}&=&(\nu\operatorname{div}\boldsymbol{v}-p)\boldsymbol{I}+2\mu\boldsymbol{D}(\boldsymbol{v}),\\ \boldsymbol{K}&=&(\alpha\Delta\rho+\beta|\nabla\rho|^2)\boldsymbol{I}+\delta(\nabla\rho\otimes\nabla\rho)+\gamma D_x^2\rho,\end{array}\right. \tag{2}$$

where $\boldsymbol{D}(\boldsymbol{v})=(\nabla\boldsymbol{v}+\nabla\boldsymbol{v}^T)/2$ is the strain tensor and $D_x^2\rho$ is the hessian matrix of the density ρ. Here, the pressure p and the coefficients $\alpha,\beta,\gamma,\delta,\mu$ and ν are functions of ρ. The special case

$$\alpha=\kappa\rho,\quad \beta=\frac{\kappa}{2},\quad \delta=-\kappa,\quad \gamma=0,$$

for some constant $\kappa>0$, corresponds precisely to Korteweg's original assumptions connected with the variational theory of Van Der Waals. In this case, the Korteweg stress tensor yields

$$\boldsymbol{K}=\frac{\kappa}{2}(\Delta\rho^2-|\nabla\rho|^2)\boldsymbol{I}-\kappa(\nabla\rho\otimes\nabla\rho). \tag{3}$$

Writing

$$\operatorname{div}\,\boldsymbol{K}=\kappa\rho\nabla(\Delta\rho)=\kappa\nabla(\rho\Delta\rho)-\kappa\nabla\rho\Delta\rho, \tag{4}$$

and incorporating $\nabla(\rho\Delta\rho)$ in the pressure term, we obtain $-\kappa\nabla\rho\Delta\rho$ as a right hand side term in the momentum equation.

The Korteweg's theory can be applied to processes of slow diffusion on miscible incompressible fluids, for example, water and glycerin. The two fluids are characterized by their reference mass density : $\bar{\rho}_1$ the density of the dilute phase and $\bar{\rho}_2$ the density of the dense phase. We need the velocity field of each constituent : $\boldsymbol{v}_1(t,\boldsymbol{x})$ and $\boldsymbol{v}_2(t,\boldsymbol{x})$, respectively. We define the volume fraction of the dilute phase $0\leq\phi(t,\boldsymbol{x})\leq 1$:

$$\phi(t,\boldsymbol{x})=\lim_{r\to 0}\frac{\text{Volume occupied at time } t \text{ by the dilute phase in } B(\boldsymbol{x},r)}{|B(\boldsymbol{x},r)|}.$$

Then, admitting that each fluid is incompressible and keeping a constant mass density, the density of the mixture is defined by

$$\rho(t,\boldsymbol{x}) = \underbrace{\bar{\rho}_2\big(1-\phi(t,\boldsymbol{x})\big)}_{:=\rho_2(t,\boldsymbol{x})} + \underbrace{\bar{\rho}_1\phi(t,\boldsymbol{x})}_{:=\rho_1(t,\boldsymbol{x})} = \bar{\rho}_2 + (\bar{\rho}_1-\bar{\rho}_2)\phi(t,\boldsymbol{x}).$$

Writing the mass conservation for the two phases, we obtain

$$\partial_t\rho + \mathrm{div}\,(\rho\boldsymbol{v}) = 0,$$

with $\rho\boldsymbol{v}(t,\boldsymbol{x}) = (\rho_2\boldsymbol{v}_2+\rho_1\boldsymbol{v}_1)(t,\boldsymbol{x})$ presents the *mean mass velocity* $\boldsymbol{v}(t,\boldsymbol{x})$, which is not divergence free, $\mathrm{div}\,\boldsymbol{v}\neq 0$. Moreover, we define the *mean volume velocity*

$$\boldsymbol{u}(t,\boldsymbol{x}) = \big(1-\phi(t,\boldsymbol{x})\big)\boldsymbol{v}_2(t,\boldsymbol{x}) + \phi(t,\boldsymbol{x})\boldsymbol{v}_1(t,\boldsymbol{x}).$$

Applying the definitions, we verify that the velocity field $\boldsymbol{u}$ is solenoidal ($\mathrm{div}\,\boldsymbol{u}=0$). According to Kazhikhov and Smagulov [10], we consider the following non-standard constraint associated to the pressure p :

$$\mathrm{div}\,\boldsymbol{v} = -\mathrm{div}\,(\lambda\nabla\ln(\rho)), \tag{5}$$

where $\lambda>0$ is a diffusion coefficient. This Fick's law (5) describes the diffusive fluxes of one fluid into the other, see also [3]. Obviously, when we set

$$\boldsymbol{v} = \boldsymbol{u} - \lambda\nabla\ln(\rho), \tag{6}$$

the relation yields (5). The mixture density ρ verifies the mass conservation and we obtain

$$\partial_t\rho + \mathrm{div}\,(\rho\boldsymbol{u}) = \mathrm{div}\,(\lambda\nabla\rho). \tag{7}$$

For the momentum equation $(1)_2$, we start by developing each term using the relation (6), in order to eliminate $\boldsymbol{v}$. After some calculations and using (4), we get

$$\begin{aligned}\partial_t\big(\rho\boldsymbol{u}\big) \;&+\; \mathrm{div}\big(\rho\boldsymbol{u}\otimes\boldsymbol{u}\big) - \lambda\,\mathrm{div}\big(\nabla\rho\otimes\boldsymbol{u}\big) - \lambda\,\mathrm{div}\big(\boldsymbol{u}\otimes\nabla\rho\big)\\ &+\; \lambda\nabla\big(\boldsymbol{u}\cdot\nabla\rho\big) + \lambda\,\mathrm{div}\big(2\mu D_x^2\ln(\rho)\big) - \mathrm{div}\big(2\mu\boldsymbol{D}(\boldsymbol{u})\big) + \nabla p\\ &-\; \lambda^2\Big(\nabla\Delta\rho - \mathrm{div}\big(\frac{\nabla\rho\otimes\nabla\rho}{\rho}\big)\Big) = \rho\boldsymbol{g} + \kappa\nabla(\rho\Delta\rho) - \kappa\nabla\rho\Delta\rho.\end{aligned} \tag{8}$$

Choosing the dynamic viscosity μ constant, as in [6], we have $\mathrm{div}\big(2\mu\boldsymbol{D}(\boldsymbol{u})\big) = \mu\boldsymbol{\Delta u}$ and $\mathrm{div}\big(2\mu D_x^2\ln(\rho)\big) = 2\mu\nabla\Delta\ln(\rho)$. Including all the gradient terms in the modified pressure

$$P = p + \lambda(\nu+2\mu)\Delta\ln(\rho) + \lambda\boldsymbol{u}\cdot\nabla\rho - \lambda^2\Delta\rho - \kappa\rho\Delta\rho.$$

Then, we obtain the Kazhikhov-Smagulov-Korteweg model in conservative form :

$$\left\{\begin{aligned}&\partial_t\big(\rho\boldsymbol{u}\big) + \mathrm{div}\,\big(\rho\boldsymbol{u}\otimes\boldsymbol{u}\big) - \lambda\,\mathrm{div}\,\big(\nabla\rho\otimes\boldsymbol{u}\big) - \lambda\,\mathrm{div}\,\big(\boldsymbol{u}\otimes\nabla\rho\big) - \mu\boldsymbol{\Delta u}\\ &\qquad\qquad +\nabla P + \lambda^2\mathrm{div}\,\big(\frac{\nabla\rho\otimes\nabla\rho}{\rho}\big) = \rho\boldsymbol{g} - \kappa\nabla\rho\Delta\rho,\\ &\partial_t\rho + \mathrm{div}\,\big(\rho\boldsymbol{u}\big) = \lambda\Delta\rho,\\ &\mathrm{div}\,\boldsymbol{u} = 0.\end{aligned}\right. \tag{9}$$

The tensorial product matrix of two vectors $\boldsymbol{a} = (a_i)_{i=1}^d$, $\boldsymbol{b} = (b_i)_{i=1}^d$ is denoted by $\boldsymbol{a} \otimes \boldsymbol{b}$ with cœfficients $(\boldsymbol{a} \otimes \boldsymbol{b})_{i,j} = a_i b_j$. Taking into account the equalities

$$\begin{aligned}\partial_t\big(\rho\boldsymbol{u}\big) + \operatorname{div}\big(\rho\boldsymbol{u}\otimes\boldsymbol{u}\big) - \lambda\operatorname{div}\big(\nabla\rho\otimes\boldsymbol{u}\big) &= \rho\partial_t\boldsymbol{u} + \rho(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \lambda(\nabla\rho\cdot\nabla)\boldsymbol{u},\\ -\lambda\operatorname{div}\big(\boldsymbol{u}\otimes\nabla\rho\big) = -\lambda(\boldsymbol{u}\cdot\nabla)\nabla\rho &= -\lambda\nabla(\boldsymbol{u}\cdot\nabla\rho) + \lambda\operatorname{div}\big(\rho\nabla\boldsymbol{u}^T\big).\end{aligned}$$

Then, denoting $\mathcal{Q}_T = (0,T)\times\Omega$, $\Sigma = (0,T)\times\Gamma$, the Kazhikhov-Smagulov-Korteweg (KSK) model can be written in $\mathcal{Q}_T$ as :

$$\begin{cases}\rho\big(\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\big) - \lambda(\nabla\rho\cdot\nabla)\boldsymbol{u} + \lambda\operatorname{div}\big(\rho\nabla\boldsymbol{u}^T\big) - \mu\boldsymbol{\Delta}\boldsymbol{u} + \nabla P\\ \qquad\qquad + \lambda^2\operatorname{div}\big(\dfrac{\nabla\rho\otimes\nabla\rho}{\rho}\big) = \rho\boldsymbol{g} - \kappa\Delta\rho\nabla\rho,\\ \partial_t\rho + \operatorname{div}\big(\rho\boldsymbol{u}\big) = \lambda\Delta\rho,\\ \operatorname{div}\boldsymbol{u} = 0.\end{cases} \tag{10}$$

The KSK model (10) is completed by the following boundary and initial conditions

$$\boldsymbol{u}(t,\boldsymbol{x}) = 0, \qquad \frac{\partial\rho}{\partial\boldsymbol{n}}(t,\boldsymbol{x}) = 0, \qquad (t,\boldsymbol{x})\in\Sigma, \tag{11}$$

$$\boldsymbol{u}(0,\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \rho(0,\boldsymbol{x}) = \rho_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega, \tag{12}$$

with the compatibility condition $\operatorname{div}\boldsymbol{u}_0 = 0$, where $\rho_0 : \Omega \to \mathbb{R}$ and $\boldsymbol{u}_0 : \Omega \to \mathbb{R}^3$ are given functions. Throughout this work, we assume the hypothesis

$$0 < m \le \rho_0(\boldsymbol{x}) \le M < +\infty, \quad \boldsymbol{x}\in\Omega. \tag{13}$$

The paper is organized as follows. In Section 2 we present the main results about (10). After some preliminary results recalled in Section 3, the proof of existence of global weak solution for (10) is given in Section 4. The conclusions are summarized in Section 5.

2. Functional setup and main results

Let us introduce the following functional spaces (see [11, 13] for their properties):

$$\begin{aligned}\mathcal{V} &= \big\{\boldsymbol{u}\in\mathcal{D}(\Omega)^3 :\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega\big\},\\ \mathbf{V} &= \big\{\boldsymbol{u}\in\mathbf{H}_0^1(\Omega) :\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega\big\},\\ \mathbf{H} &= \big\{\boldsymbol{u}\in\mathbf{L}^2(\Omega) :\ \operatorname{div}\boldsymbol{u} = 0 \text{ in } \Omega,\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\big\},\\ H_N^s &= \left\{\rho\in H^s(\Omega) :\ \frac{\partial\rho}{\partial\boldsymbol{n}} = 0 \text{ on } \Gamma,\ \int_\Omega \rho(\boldsymbol{x})\,d\boldsymbol{x} = \int_\Omega \rho_0(\boldsymbol{x})\,d\boldsymbol{x}\right\}, \quad s\ge 2.\end{aligned}$$

The spaces $\mathbf{V}$ and $\mathbf{H}$ are the closures of $\mathcal{V}$ in $\mathbf{H}_0^1(\Omega)$ and $\mathbf{L}^2(\Omega)$, respectively.

Let us recall the definition of weak solution for the KSK model (10). Such class of solutions can be found in [1] for Kazhikhov-Smagulov type models and in [13] for the incompressible Navier-Stokes equations.

Definition 2.1 *A pair of functions $(\boldsymbol{u},\rho)$ is called a weak solution of problem (10),(11),(12) on Ω if and only if the following assumptions are satisfied :*

1) $\boldsymbol{u} \in L^\infty\big(0,T;\boldsymbol{H}\big) \cap L^2\big(0,T;\boldsymbol{V}\big)$, $\rho \in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^2_N\big)$ *and*

$$0 < m \le \rho(t,\boldsymbol{x}) \le M < +\infty, \text{ a.e. } (t,\boldsymbol{x}) \in \mathcal{Q}_T.$$

2) For all $\boldsymbol{\phi} \in \mathcal{C}^1\big([0,T];\boldsymbol{V}\big)$ *such that* $\boldsymbol{\phi}(T,.) = 0$*, one has :*

$$\int_0^T \Big\{ -\big(\boldsymbol{u}, \rho\partial_t\boldsymbol{\phi} + ((\rho\boldsymbol{u} - \lambda\nabla\rho)\cdot\nabla)\boldsymbol{\phi}\big) + \mu\big(\nabla\boldsymbol{u}, \nabla\boldsymbol{\phi}\big) - \lambda\big(\rho\nabla\boldsymbol{u}^T, \nabla\boldsymbol{\phi}\big)\Big\}dt$$
$$-\lambda^2\int_0^T \big(\frac{1}{\rho}\nabla\rho\otimes\nabla\rho, \nabla\boldsymbol{\phi}\big)dt = \int_0^T \big(\rho\boldsymbol{g} - \kappa\Delta\rho\nabla\rho, \boldsymbol{\phi}\big)dt + \big(\rho_0\boldsymbol{u}_0, \boldsymbol{\phi}(0)\big). \tag{14}$$

3) For all $\varphi \in \mathcal{C}^1\big([0,T];H^1(\Omega)\big)$ *such that* $\varphi(T,.) = 0$*, one has :*

$$\int_0^T \Big\{\big(\boldsymbol{u}\cdot\nabla\rho, \varphi\big) + \lambda\big(\nabla\rho, \nabla\varphi\big) - \big(\rho, \partial_t\varphi\big)\Big\}dt = \big(\rho_0, \varphi(0)\big). \tag{15}$$

REMARK. — The pressure P associated with the weak solution $(\boldsymbol{u}, \rho)$ can be obtained using (14) and the Rham's lemma [13].

We present the aim of this work about the Kazhikhov-Smagulov-Korteweg model (10). Under some assumption on the coefficients λ, μ, κ, we prove the global existence of weak solution of (10) for arbitrary initial data and external force field. Our main result reads :

Theorem 2.2 *Let* $\boldsymbol{u}_0 \in \boldsymbol{H}$*,* $\rho_0 \in H^1(\Omega)$ *satisfy (13),* $T > 0$ *and* $\boldsymbol{g} \in L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big)$*. If* $\dfrac{\lambda}{\mu}\max(1, \dfrac{\lambda^2}{\kappa})$ *is sufficiently small, then there exists a weak solution* $(\boldsymbol{u}, \rho)$ *of (10) global in time such that*

$$\boldsymbol{u} \in L^\infty\big(0,T;\boldsymbol{H}\big) \cap L^2\big(0,T;\boldsymbol{V}\big),$$
$$\rho \in L^\infty\big(0,T;H^1(\Omega)\big) \cap L^2\big(0,T;H^2_N\big),$$

with finite and uniformly bounded energy such that $\forall t \le T$*,*

$$\| \sqrt{\rho(t)}\boldsymbol{u}(t) \|^2_{L^2(\Omega)} + \kappa \| \nabla\rho(t) \|^2_{L^2(\Omega)} + \int_0^t \Big(\frac{\mu}{2} \| \nabla\boldsymbol{u}(s) \|^2_{L^2(\Omega)} + \kappa\lambda \| \Delta\rho(s) \|^2_{L^2(\Omega)}\Big)ds$$
$$\le \| \sqrt{\rho_0}\boldsymbol{u}_0 \|^2_{L^2(\Omega)} + \kappa \| \nabla\rho_0 \|^2_{L^2(\Omega)} + \frac{CM^2}{\mu}\int_0^T \| \boldsymbol{g}(s) \|^2_{L^2(\Omega)} ds.$$

3. Preliminary results

Given the initial density ρ_0 and the velocity field $\boldsymbol{u}$, we find the density ρ as solution of the following Neumann problem :

$$\left\{\begin{array}{ll} \partial_t\rho + \boldsymbol{u}\cdot\nabla\rho = \lambda\,\Delta\rho & \text{in } \mathcal{Q}_T, \\ \rho(0,\boldsymbol{x}) = \rho_0(\boldsymbol{x}) & \text{in } \Omega, \\ \dfrac{\partial\rho}{\partial\boldsymbol{n}} = 0 & \text{on } \Sigma. \end{array}\right. \tag{16}$$

The density ρ satisfies the maximum principle. This result is classical (see [1]).

Proposition 3.1 *If $(\boldsymbol{u},\rho)$ is a weak solution of (10), then*

$$0 < m \le \rho(t,\boldsymbol{x}) \le M < +\infty \qquad \textit{a.e. } (t,\boldsymbol{x}) \in \mathcal{Q}_T. \tag{17}$$

Proposition 3.2 *Let $\rho_0 \in H^1(\Omega)$ verify (13) and $\boldsymbol{u} \in \mathcal{C}\big([0,T]; \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)\big)$. Then there exists a unique solution ρ of (16) such that*

$$\rho \in L^\infty\big(0,T; H^1(\Omega)\big) \cap L^2\big(0,T; H^2_N\big).$$

Moreover, we have

$$\sup_{0\le t\le T} \| \rho(t) \|^2_{L^2(\Omega)} \le \| \rho_0 \|^2_{H^1(\Omega)}, \tag{18}$$

$$\int_0^T \| \nabla\rho(t) \|^2_{L^2(\Omega)} \, dt \le \frac{1}{2\lambda} \| \rho_0 \|^2_{H^1(\Omega)}, \tag{19}$$

$$\sup_{0\le t\le T} \| \nabla\rho(t) \|^2_{L^2(\Omega)} \le C_\lambda \| \rho_0 \|^2_{H^1(\Omega)} \big(1 + \sup_{0\le t\le T} \| \boldsymbol{u}(t) \|^2_{L^\infty(\Omega)} \big), \tag{20}$$

$$\int_0^T \| \Delta\rho(t) \|^2_{L^2(\Omega)} \, dt \le \frac{C_\lambda}{\lambda} \| \rho_0 \|^2_{H^1(\Omega)} \big(1 + \sup_{0\le t\le T} \| \boldsymbol{u}(t) \|^2_{L^\infty(\Omega)} \big), \tag{21}$$

where C_λ is a positive constant depending only on λ.

Given $\rho_0 \in H^1(\Omega)$ satisfying (13) and $\boldsymbol{u} \in \mathcal{C}\big([0,T]; \mathbf{V} \cap \mathbf{H}^2(\Omega)\big)$, let ρ the solution obtained by Proposition 3.2. Therefore, it is clear that the following map is well defined

$$\mathcal{S} : \mathcal{C}\big([0,T]; \mathbf{V} \cap \mathbf{H}^2(\Omega)\big) \longrightarrow L^\infty\big(0,T; H^1(\Omega)\big) \cap L^2\big(0,T; H^2_N\big),$$

such that $\rho = \mathcal{S}\boldsymbol{u}$ is well defined.

Proposition 3.3 *Let $\rho_0 \in H^1(\Omega)$ verify (13) and $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{C}\big([0,T]; \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)\big)$. Set $\rho = \rho_1 - \rho_2 = \mathcal{S}\boldsymbol{u}_1 - \mathcal{S}\boldsymbol{u}_2$ and $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$, we have the following estimates :*

$$\sup_{0\le t\le T} \| \rho(t) \|^2_{L^2(\Omega)} + \lambda \int_0^T \| \nabla\rho(t) \|^2_{L^2(\Omega)} \, dt \le \frac{M^2}{\lambda} T \sup_{0\le t\le T} \| \boldsymbol{u}(t) \|^2_{L^2(\Omega)}, \tag{22}$$

$$\begin{aligned} &\sup_{0\le t\le T} \| \nabla\rho(t) \|^2_{L^2(\Omega)} + \lambda \int_0^T \| \Delta\rho(t) \|^2_{L^2(\Omega)} \, dt \\ &\le \frac{2T}{\lambda} \sup_{0\le t\le T} \| \nabla\rho_1 \|^2_{L^2(\Omega)} \sup_{0\le t\le T} \| \boldsymbol{u} \|^2_{L^\infty(\Omega)} + \frac{2M^2T}{\lambda^3} \sup_{0\le t\le T} \| \boldsymbol{u}_2 \|^2_{L^\infty(\Omega)} \sup_{0\le t\le T} \| \boldsymbol{u} \|^2_{L^2(\Omega)} . \end{aligned} \tag{23}$$

We recall that there exists an orthonormal basis of $\mathbf{L}^2(\Omega)$ defined by

$$\begin{aligned} &\boldsymbol{\omega}_k \in \mathbf{V} \cap \mathbf{H}^2(\Omega) \\ &-\mathbb{P}\boldsymbol{\Delta}\boldsymbol{\omega}_k = \lambda_k \, \boldsymbol{\omega}_k \qquad \text{on } \Omega, \end{aligned}$$

where $\mathbb{P}$ is the orthogonal projection operator of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. For any $n \in \mathbb{N}^*$, we define by $\mathbf{X_n}$ the finite dimensional subspace of $\mathbf{H}$ such that

$$\mathbf{X_n} = \mathcal{V}\text{ect}\{\boldsymbol{\omega}_k, k = 1, \ldots, n\},$$

and we consider the orthogonal projection $\mathbb{P}_n : \mathbf{L}^2(\Omega) \to \mathbf{X_n}$ defined by

$$\forall \boldsymbol{w} \in \mathbf{H}, \qquad \big(\mathbb{P}_n \boldsymbol{w},\ \boldsymbol{v}\big) \ = \ \big(\boldsymbol{w},\ \boldsymbol{v}\big), \qquad \forall \boldsymbol{v} \in \mathbf{X_n}. \tag{24}$$

As in [5], we introduce a family of operators $\mathcal{M}[\rho] : \mathbf{X_n} \longrightarrow \mathbf{X_n}$ defined by

$$\big(\mathcal{M}[\rho]\boldsymbol{v},\ \boldsymbol{\omega}\big) \ = \ \int_\Omega \rho\, \boldsymbol{v} \cdot \boldsymbol{\omega}\, d\boldsymbol{x} \qquad \text{for all } \boldsymbol{v}, \boldsymbol{\omega} \in \mathbf{X_n}. \tag{25}$$

If $\rho \in L^\infty(\Omega)$, then $\mathcal{M}[\rho]$ is well defined. Moreover, let $m > 0$, we set

$$\mathcal{D} \ = \ \Big\{ \rho \in L^\infty(\Omega);\ \ \rho(\boldsymbol{x}) \geq m > 0 \Big\}.$$

Proposition 3.4 *$\mathcal{M}[\rho]$ is one-to-one and its inverse verifies*

$$\| \,\mathcal{M}[\rho]^{-1} \,\|_{\mathcal{L}(\mathbf{X_n}, \mathbf{X_n})} \ \leq \ \big(\inf_{\boldsymbol{x}\in\Omega} \rho(\boldsymbol{x})\big)^{-1} \qquad \forall \rho \in \mathcal{D}, \tag{26}$$

$$\| \,\mathcal{M}[\rho_1]^{-1} - \mathcal{M}[\rho_2]^{-1} \,\|_{\mathcal{L}(\mathbf{X_n}, \mathbf{X_n})} \ \leq \ \frac{C_n}{m^2} \ \| \,\rho_1 - \rho_2 \,\|_{L^2(\Omega)} \quad \forall \rho_1, \rho_2 \in \mathcal{D}, \tag{27}$$

where C_n is a constant depending on the dimension of the space $\mathbf{X_n}$.

4. Proof of Theorem 2.2

4.1. Faedo-Galerkin method

We are looking for the approximate solutions

$$(\boldsymbol{u}_n, \rho_n) \in \mathcal{C}\big([0,T]; \mathbf{X_n}\big) \times \mathcal{C}\big([0,T]; H^1(\Omega) \cap H^2_N\big)$$

satisfying

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \partial_t\big(\rho_n \boldsymbol{u}_n\big) \cdot \boldsymbol{v} d\boldsymbol{x} + \int_\Omega \rho_n (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n \cdot \boldsymbol{v} d\boldsymbol{x} - \lambda \int_\Omega (\nabla\rho_n \cdot \nabla)\boldsymbol{u}_n \cdot \boldsymbol{v} d\boldsymbol{x} \\ \displaystyle + \int_\Omega (\boldsymbol{u}_n \cdot \nabla\rho_n)\, \boldsymbol{u}_n \cdot \boldsymbol{v} d\boldsymbol{x} - \lambda \int_\Omega \Delta\rho_n \boldsymbol{u}_n \cdot \boldsymbol{v} d\boldsymbol{x} - \mu \int_\Omega \boldsymbol{\Delta}\boldsymbol{u}_n \cdot \boldsymbol{v} d\boldsymbol{x} \\ \displaystyle + \lambda \int_\Omega \mathrm{div}\, \big(\rho_n \nabla \boldsymbol{u}_n^T\big) \cdot \boldsymbol{v} d\boldsymbol{x} + \lambda^2 \int_\Omega \mathrm{div}\, \Big(\frac{\nabla\rho_n \otimes \nabla\rho_n}{\rho_n}\Big) \cdot \boldsymbol{v} d\boldsymbol{x} \\ \displaystyle = \int_\Omega \rho_n \boldsymbol{g} \cdot \boldsymbol{v} d\boldsymbol{x} - \kappa \int_\Omega \Delta\rho_n \nabla\rho_n \cdot \boldsymbol{v} d\boldsymbol{x}, \qquad \forall \boldsymbol{v} \in \mathbf{X_n}, \\ \displaystyle \int_\Omega \partial_t\big(\rho_n\big)\, \eta\, d\boldsymbol{x} + \int_\Omega \boldsymbol{u}_n \cdot \nabla\rho_n\, \eta\, d\boldsymbol{x} = \lambda \int_\Omega \Delta\rho_n\, \eta\, d\boldsymbol{x}, \qquad \forall \eta \in H^1(\Omega), \\ \boldsymbol{u}_n(0) = \boldsymbol{u}_{0n} = \mathbb{P}_n \boldsymbol{u}_0, \\ \rho_n(0) = \rho_0. \end{array} \right. \tag{28}$$

We set

$$\begin{aligned} \mathcal{N}[\boldsymbol{u}_n, \rho_n] = &-\big((\rho_n \boldsymbol{u}_n - \lambda\nabla\rho_n) \cdot \nabla\big)\boldsymbol{u}_n - (\boldsymbol{u}_n \cdot \nabla\rho_n)\boldsymbol{u}_n + \lambda\Delta\rho_n \boldsymbol{u}_n \\ &+ \mu\boldsymbol{\Delta}\boldsymbol{u}_n - \lambda\, \mathrm{div}\big(\rho_n \nabla\boldsymbol{u}_n^T\big) - \lambda^2 \mathrm{div}\Big(\frac{\nabla\rho_n \otimes \nabla\rho_n}{\rho_n}\Big) - \kappa\Delta\rho_n\nabla\rho_n + \rho_n \boldsymbol{g}. \end{aligned} \tag{29}$$

Taking $(28)_1$ with $\boldsymbol{v} = \boldsymbol{\omega}_k$, for $k = 1, \dots, n,$ and integrating in time between 0 and $t \leq T$, the solution $\boldsymbol{u}_n$ verifies the following integral equations for $k = 1, \dots, n$:

$$\int_\Omega \rho_n(t)\boldsymbol{u}_n(t) \cdot \boldsymbol{\omega}_k \, d\boldsymbol{x} \ = \int_\Omega \boldsymbol{q}_0 \cdot \boldsymbol{\omega}_k \, d\boldsymbol{x} \ + \int_0^t \int_\Omega \mathcal{N}[\boldsymbol{u}_n, \rho_n] \cdot \boldsymbol{\omega}_k \, d\boldsymbol{x} \, ds, \tag{30}$$

where $\rho_n = \mathcal{S}\boldsymbol{u}_n$ and $\boldsymbol{q}_0 = \rho_0 \boldsymbol{u}_{0n}$. Thanks to (24) and (25), we rewrite (30) as follows :

$$\Big(\mathcal{M}[\rho_n(t)]\boldsymbol{u}_n(t), \, \boldsymbol{\omega}_k\Big) = \Big(\mathbb{P}_n \boldsymbol{q}_0, \, \boldsymbol{\omega}_k\Big) + \Big(\mathbb{P}_n \int_0^t \mathcal{N}[\boldsymbol{u}_n(s), \rho_n(s)] \, ds, \, \boldsymbol{\omega}_k\Big),$$

for $k = 1, \dots, n$. Since $\mathcal{M}[\rho_n]$ is invertible, then the resulting equation reads

$$\boldsymbol{u}_n \in \mathcal{C}\big([0,T]; \mathbf{X_n}\big), \quad \boldsymbol{u}_n(t) = \mathcal{M}[\rho_n(t)]^{-1}\mathbb{P}_n\Big(\boldsymbol{q}_0 + \int_0^t \mathcal{N}[\boldsymbol{u}_n(s), \rho_n(s)] \, ds\Big). \tag{31}$$

Hence, $\boldsymbol{u}_n$ appears as a fixed point of a suitable functional Ψ

$$\begin{array}{rccc} \Psi : & \mathcal{C}\big([0,T]; \mathbf{X_n}\big) & \longrightarrow & \mathcal{C}\big([0,T]; \mathbf{X_n}\big) \\ & \boldsymbol{u}_n & \longmapsto & \Psi\big(\boldsymbol{u}_n\big) \end{array}$$

defined by

$$\Psi\big(\boldsymbol{u}_n\big)(t) = \mathcal{M}[\rho_n(t)]^{-1}\mathbb{P}_n\Big(\boldsymbol{q}_0 + \int_0^t \mathcal{N}[\boldsymbol{u}_n(s), \rho_n(s)] \, ds\Big), \qquad \text{for all } t \in [0,T].$$

Let $\mathbf{X_T}$ be the Banach space $\mathcal{C}\big([0,T]; \mathbf{X_n}\big)$ endowed with the norm

$$\| \boldsymbol{u}_n \|_{\mathbf{X_T}} = \sup_{0 \leq t \leq T} \| \boldsymbol{u}_n(t) \|_{\mathbf{L}^2(\Omega)} .$$

In order to apply the Banach fixed point theorem, we establish some uniform estimates for Ψ. With Propositions 3.2, 3.3 and 3.4 in mind, we have the following :

Proposition 4.1 *There exists a constant $C > 0$ depending on n, λ, μ, κ, M, m, $\| \rho_0 \|_{H^1(\Omega)}$, $\| \boldsymbol{g} \|_{L^2(0,T;L^2(\Omega))}$, such that for all $\boldsymbol{u}_n \in \mathbf{X_T}$,*

$$\| \Psi\big(\boldsymbol{u}_n\big) \|_{\mathbf{X_T}} \leq \ \frac{M}{m} \, \| \boldsymbol{u}_0 \|_{L^2(\Omega)} + C \max(T, T^{\frac{1}{4}})\Big(1 + \| \boldsymbol{u}_n \|^2_{\mathbf{X_T}} \ \Big), \tag{32}$$

and for all $\boldsymbol{u}_n^1, \boldsymbol{u}_n^2 \in \mathbf{X_T}$,

$$\begin{aligned} \| \Psi\big(\boldsymbol{u}_n^1\big) - \Psi\big(\boldsymbol{u}_n^2\big) \|_{\mathbf{X_T}} \ &\leq \ C \max(T, T^{\frac{1}{4}})\Big(1 + \| \boldsymbol{u}_0 \|_{L^2(\Omega)} \\ &\quad + \ \| \boldsymbol{u}_n^1 \|^2_{\mathbf{X_T}} + \| \boldsymbol{u}_n^2 \|^2_{\mathbf{X_T}} \ \Big) \, \| \boldsymbol{u}_n^1 - \boldsymbol{u}_n^2 \|_{\mathbf{X_T}} \ . \end{aligned} \tag{33}$$

At this stage, we set $R = 2\dfrac{M}{m} \, \| \boldsymbol{u}_0 \|_{L^2(\Omega)}$ and $\mathcal{B}_R^T = \Big\{ \boldsymbol{u} \in \mathbf{X_T}, \ \| \boldsymbol{u} \|_{\mathbf{X_T}} \leq R \Big\}$.

Proposition 4.2 *There exists $T_n \in]0,1[$ small enough and $\boldsymbol{u}_n \in \mathcal{B}_R^{T_n}$ such that*

$$\boldsymbol{u}_n = \Psi(\boldsymbol{u}_n).$$

Proof. Let $0 < T_n < 1$ such that

$$\max\Big(CT_n^{\frac{1}{4}}\big[R + \frac{1}{R}\big],\ CT_n^{\frac{1}{4}}\big[1 + \| \boldsymbol{u}_0 \|_{L^2(\Omega)} + 2R^2\big]\Big) \leq \frac{1}{2}.$$

Thanks to Proposition 4.1, we verify that Ψ is a contraction mapping on $\mathcal{B}_R^{T_n}$ and we conclude the existence of a unique fixed point of Ψ. ■

It is clear that $\boldsymbol{u}_n$ the fixed point of Ψ, obtained in Proposition 4.2, implies that $(\boldsymbol{u}_n, \rho_n = \mathcal{S}\boldsymbol{u}_n)$ is a local solution of the Galerkin approximate problem (28). Now, we will prove that this local solution is in fact a global one. For this, we establish some uniform estimates for $(\boldsymbol{u}_n, \rho_n)$ with respect to time.

Proposition 4.3 *If* $\dfrac{\lambda}{\mu}\max(1, \dfrac{\lambda^2}{\kappa})$ *small enough, there exists a constant* $C > 0$ *depending on* $\rho_0, \boldsymbol{u}_0, \boldsymbol{g}, M, \mu, \kappa$*, such that for all* $t \in [0, T_n)$

$$m \ \| \boldsymbol{u}_n(t) \|^2_{L^2(\Omega)} + \frac{\mu}{2}\int_0^t \| \nabla \boldsymbol{u}_n(s) \|^2_{L^2(\Omega)} \, ds \ \leq \ C, \tag{34}$$

$$\kappa \ \| \nabla \rho_n(t) \|^2_{L^2(\Omega)} + \kappa\lambda \int_0^t \| \Delta \rho_n(s) \|^2_{L^2(\Omega)} \, ds \ \leq \ C. \tag{35}$$

Evidently, thanks to the previous Proposition 4.3, we have the following :

Corollary 4.4 $(\boldsymbol{u}_n, \rho_n)$ *is a global solution of (28) and for all* $T > 0$,

$$(\boldsymbol{u}_n)_n \ \text{ is bounded in } \ L^\infty\big(0, T; \boldsymbol{H}\big) \cap L^2\big(0, T; \boldsymbol{V}\big), \tag{36}$$

$$(\rho_n)_n \ \text{ is bounded in } \ L^\infty\big(0, T; H^1(\Omega)\big) \cap L^2\big(0, T; H^2_N\big). \tag{37}$$

4.2. Uniform estimates for time derivatives

In this section, we establish uniform estimates for time derivatives $\partial_t \rho_n$ and $\partial_t \boldsymbol{u}_n$.

Proposition 4.5 *Let* $T > 0$*. The sequence* $(\partial_t \rho_n)_n$ *is bounded in* $L^{4/3}\big(0, T; L^2(\Omega)\big)$.

Proof. Taking the L^2-norm of $\partial_t \rho_n$. Applying the Hölder and Gagliardo-Nirenberg inequalities and the inequality : $\| \nabla \rho \|_{L^4(\Omega)} \leq C_0 \| \rho \|^{1/2}_{L^\infty(\Omega)} \| \Delta \rho \|^{1/2}_{L^2(\Omega)}$, we get

$$\| \partial_t \rho_n \|_{L^2(\Omega)} \leq \lambda \| \Delta \rho_n \|_{L^2(\Omega)} + C \| \boldsymbol{u}_n \|^{1/4}_{L^2(\Omega)} \| \nabla \boldsymbol{u}_n \|^{3/4}_{L^2(\Omega)} \| \rho_n \|^{1/2}_{L^\infty(\Omega)} \| \Delta \rho_n \|^{1/2}_{L^2(\Omega)} \ .$$

By the uniform estimate (34) and (17), we get

$$\| \partial_t \rho_n \|_{L^2(\Omega)} \leq \ \lambda \ \| \Delta \rho_n \|_{L^2(\Omega)} + C \ \| \nabla \boldsymbol{u}_n \|^{3/4}_{L^2(\Omega)} \| \Delta \rho_n \|^{1/2}_{L^2(\Omega)} \ . \tag{38}$$

Next, applying the Young inequality $ab \leq \dfrac{1}{2}(a^2 + b^2)$ in (38), we get

$$\| \partial_t \rho_n \|_{L^2(\Omega)} \leq \ \lambda \ \| \Delta \rho_n \|_{L^2(\Omega)} + C \ \| \nabla \boldsymbol{u}_n \|^{3/2}_{L^2(\Omega)} \ .$$

Thanks to the uniform time estimates (34) and (35), we deduce that $\| \ \partial_t \rho_n \ \|_{L^2(\Omega)}$ is bounded in $L^{4/3}\big(0, T\big)$. ■

Now, by following [1], we establish an estimation of the fractional time derivative of $\boldsymbol{u}_n$.

Proposition 4.6 *Let $0 < \delta < T$ such that*

$$\int_0^{T-\delta} \| \boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t) \|^2_{L^2(\Omega)} \, dt \leq C\, \delta^{\frac{1}{4}}, \tag{39}$$

where C a constant independent of n and δ.

Proof. For all functions $\boldsymbol{\phi} \in \mathbf{X_T}$, the approximate solution $(\boldsymbol{u}_n, \rho_n)$ verifies :

$$\begin{aligned} &\frac{d}{d\tau}\int_\Omega \rho_n \boldsymbol{u}_n \cdot \boldsymbol{\phi}\, d\boldsymbol{x} - \int_\Omega \rho_n \boldsymbol{u}_n \cdot \frac{\partial \boldsymbol{\phi}}{\partial \tau}\, d\boldsymbol{x} - \int_\Omega \rho_n (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{\phi} \cdot \boldsymbol{u}_n \, d\boldsymbol{x} \\ &+ \mu \int_\Omega \nabla \boldsymbol{u}_n : \nabla \boldsymbol{\phi}\, d\boldsymbol{x} + \lambda \int_\Omega (\nabla \rho_n \cdot \nabla)\boldsymbol{\phi} \cdot \boldsymbol{u}_n \, d\boldsymbol{x} - \lambda \int_\Omega \rho_n \nabla \boldsymbol{u}_n^T : \nabla \boldsymbol{\phi}\, d\boldsymbol{x} \\ &- \lambda^2 \int_\Omega \frac{\nabla \rho_n \otimes \nabla \rho_n}{\rho_n} : \nabla \boldsymbol{\phi}\, d\boldsymbol{x} = \int_\Omega \rho_n \boldsymbol{g} \cdot \boldsymbol{\phi}\, d\boldsymbol{x} - \kappa \int_\Omega \Delta \rho_n \nabla \rho_n \cdot \boldsymbol{\phi}\, d\boldsymbol{x}. \end{aligned} \tag{40}$$

Integrating (40) with respect to τ between t and $t+\delta$, and taking $\boldsymbol{\phi} = \boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)$

$$\begin{aligned} &\int_\Omega \big[\rho_n(t+\delta)\, \boldsymbol{u}_n(t+\delta) - \rho_n(t)\, \boldsymbol{u}_n(t)\big]\big[\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big]\, d\boldsymbol{x} \\ &= \int_t^{t+\delta}\int_\Omega \Big(\rho_n(\tau)\, \boldsymbol{g}(\tau) - \kappa\, \Delta\rho_n(\tau)\, \nabla \rho_n(\tau)\Big) \cdot \big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big)\, d\boldsymbol{x}\, d\tau \\ &+ \int_t^{t+\delta}\int_\Omega \Big(\big(\rho_n(\tau)\boldsymbol{u}_n(\tau) - \lambda \nabla \rho_n(\tau)\big) \cdot \nabla\Big)\big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big) \cdot \boldsymbol{u}_n(\tau)\, d\boldsymbol{x}\, d\tau \\ &- \int_t^{t+\delta}\int_\Omega \Big(\mu\, \nabla \boldsymbol{u}_n(\tau) - \lambda\, \rho_n(\tau) \nabla \boldsymbol{u}_n^T(\tau)\Big) : \nabla\big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big)\, d\boldsymbol{x}\, d\tau \\ &+ \lambda^2 \int_t^{t+\delta}\int_\Omega \frac{\nabla \rho_n(\tau) \otimes \nabla \rho_n(\tau)}{\rho_n(\tau)} : \nabla\big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big)\, d\boldsymbol{x}\, d\tau. \end{aligned} \tag{41}$$

Using the following identity

$$\rho_n(t+\delta)\, \boldsymbol{u}_n(t+\delta) - \rho_n(t)\, \boldsymbol{u}_n(t) = \rho_n(t+\delta)\big[\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big] + \big[\rho_n(t+\delta) - \rho_n(t)\big]\boldsymbol{u}_n(t),$$

then, (41) becomes

$$\begin{aligned} &\| \sqrt{\rho_n(t+\delta)}\big[\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big] \|^2_{L^2(\Omega)} \\ &= -\int_\Omega \big[\rho_n(t+\delta) - \rho_n(t)\big]\big[\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big] \cdot \boldsymbol{u}_n(t)\, d\boldsymbol{x} \\ &\quad + \int_t^{t+\delta}\int_\Omega \Big(\rho_n(\tau)\, \boldsymbol{g}(\tau) - \kappa\, \Delta\rho_n(\tau)\, \nabla \rho_n(\tau)\Big) \cdot \big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big)\, d\boldsymbol{x}\, d\tau \\ &\quad + \int_t^{t+\delta}\int_\Omega \Big(\big(\rho_n(\tau)\boldsymbol{u}_n(\tau) - \lambda \nabla \rho_n(\tau)\big) \cdot \nabla\Big)\big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big) \cdot \boldsymbol{u}_n(\tau)\, d\boldsymbol{x}\, d\tau \\ &\quad - \int_t^{t+\delta}\int_\Omega \Big(\mu\, \nabla \boldsymbol{u}_n(\tau) - \lambda\, \rho_n(\tau) \nabla \boldsymbol{u}_n^T(\tau)\Big) : \nabla\big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big)\, d\boldsymbol{x}\, d\tau \\ &\quad + \lambda^2 \int_t^{t+\delta}\int_\Omega \frac{\nabla \rho_n(\tau) \otimes \nabla \rho_n(\tau)}{\rho_n(\tau)} : \nabla\big(\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big)\, d\boldsymbol{x}\, d\tau \\ &= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t) + I_7(t) + I_8(t). \end{aligned} \tag{42}$$

Let us estimate $I_1(t)$. Applying the Hölder inequality, we get

$$|I_1(t)| \leq \| \rho_n(t+\delta) - \rho_n(t) \|_{L^2(\Omega)} \| \boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t) \|_{L^4(\Omega)} \| \boldsymbol{u}_n(t) \|_{L^4(\Omega)} .$$

In particular, we write

$$\rho_n(t+\delta) - \rho_n(t) = \int_t^{t+\delta} \frac{\partial \rho_n}{\partial \tau}\, d\tau.$$

Using the Hölder and Young inequalities and the embedding $H^1(\Omega) \subset L^4(\Omega)$, we obtain

$$|I_1(t)| \;\le\; C\delta^{\frac{1}{4}}\Big(\int_t^{t+\delta} \| \frac{\partial \rho_n}{\partial \tau} \|^{\frac{4}{3}}_{L^2(\Omega)}\, d\tau\Big)^{\frac{3}{4}} \Big(\| \nabla \boldsymbol{u}_n(t+\delta) \|^2_{L^2(\Omega)} + \| \nabla \boldsymbol{u}_n(t) \|^2_{L^2(\Omega)} \Big).$$

In the same way, we verify the following estimations :

$$|I_2(t)| \;\le\; C\delta^{\frac{1}{2}}\Big(\int_t^{t+\delta} \| \boldsymbol{g}(\tau) \|^2_{L^2(\Omega)}\, d\tau\Big)^{\frac{1}{2}} \Big(\| \nabla \boldsymbol{u}_n(t+\delta) \|^2_{L^2(\Omega)} + \| \nabla \boldsymbol{u}_n(t) \|^2_{L^2(\Omega)} \Big),$$

$$|I_3(t)| \;\le\; C\delta^{\frac{1}{4}}\Big(\int_t^{t+\delta} \| \Delta\rho_n(\tau) \|^2_{L^2(\Omega)}\, d\tau\Big)^{\frac{3}{4}} \Big(\| \nabla \boldsymbol{u}_n(t+\delta) \|^2_{L^2(\Omega)} + \| \nabla \boldsymbol{u}_n(t) \|^2_{L^2(\Omega)} \Big).$$

Similarly, one can obtain the desired estimates of $I_j(t)$ terms, for $j = 4, \ldots, 8$.
At last, if we choose $0 < \delta < 1$ and taking into account Propositions 4.3 and 4.5, then by gathering together all the above estimates, we rewrite (42) as follows :

$$\| \sqrt{\rho_n(t+\delta)}\big[\boldsymbol{u}_n(t+\delta) - \boldsymbol{u}_n(t)\big] \|^2_{L^2(\Omega)} \le C\delta^{\frac{1}{4}}\Big(\| \nabla \boldsymbol{u}_n(t+\delta) \|^2_{L^2(\Omega)} + \| \nabla \boldsymbol{u}_n(t) \|^2_{L^2(\Omega)} \Big).$$

Thanks to the lower bound of ρ_n and Proposition 4.3, we finish the proof. ■

4.3. The existence of solution $(\boldsymbol{u}, \rho)$

The final step to complete this study is to employ the previous uniform estimates in order to pass to the limit in the approximate problem (28). When $n \to +\infty$, we have

$$\boldsymbol{u}_{0n} \longrightarrow \boldsymbol{u}_0 \text{ in } \mathbf{H} \text{ strongly.}$$

Thanks to (36) and (37), choosing the subsequences $(\boldsymbol{u}_n)_n$ and $(\rho_n)_n$ such that

$$\begin{array}{lllll} \boldsymbol{u}_n & \longrightarrow & \boldsymbol{u} & \text{in } L^2\big(0,T;\mathbf{V}\big) & \text{weakly,} \\ \boldsymbol{u}_n & \longrightarrow & \boldsymbol{u} & \text{in } L^\infty\big(0,T;\mathbf{H}\big) & \text{weakly-star,} \end{array}$$

and

$$\begin{array}{lllll} \rho_n & \longrightarrow & \rho & \text{in } L^2\big(0,T;H^2_N\big) & \text{weakly,} \\ \rho_n & \longrightarrow & \rho & \text{in } L^\infty\big(0,T;H^1(\Omega)\big) & \text{weakly-star,} \\ \partial_t\,\rho_n & \longrightarrow & \partial_t\,\rho & \text{in } L^{4/3}\big(0,T;L^2(\Omega)\big) & \text{weakly.} \end{array}$$

We are able to pass to the limit in the linear terms of (28), thanks to these above convergence results. Now, to ensure the passage to the limit in the nonlinear terms of (28), it is necessary to use the following strong convergence :

Proposition 4.7 *There exists a subsequence $(\boldsymbol{u}_n, \rho_n)_n$ which converges strongly to $(\boldsymbol{u}, \rho)$ in $L^2\big(0,T;\boldsymbol{L}^2(\Omega)\big) \times L^2\big(0,T;H^1(\Omega)\big)$. Moreover, $(\boldsymbol{u}, \rho)$ is a weak solution of (10).*

Proof. Applying some compactness theorems [13, Chap.3, Theorem 2.1] for ρ_n and [12, Theorem 5] for $\boldsymbol{u}_n$ and using Propositions 4.5 and 4.6, we get to the desired result. ■

5. Conclusions

In this paper, we study the system of PDEs derived from the compressible Navier-Stokes equations with presence of a specific Korteweg stress tensor, called the Kazhikhov-Smagulov-Korteweg (KSK) model. We arrive at verify the existence of a weak solution $(\boldsymbol{u}, \rho)$ of the KSK model (10) global in time with finite and uniformly bounded energy. Then, we conclude the proof of Theorem 2.2, the main result of this paper.

6. References

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