Theoretical Analysis of a Water Wave Model using the Diffusive Approach

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ABSTRACT. In this paper, we theoretically study the water wave model with a nonlocal viscous term

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + u u_x = \nu u_{xx},$$

where the Riemann-Liouville half-order derivative $\frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds$ is represented with a diffusive realization.

RÉSUMÉ. Dans cet article, nous étudions théoriquement le modèle visqueux asymptotique

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + u u_x = \nu u_{xx},$$

où la demi-dérivée de Riemann-Liouville $\frac{1}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds$ est représentée à l'aide d'une réalisation diffusive.

KEYWORDS : nonlocal viscous model, Riemann-Liouville half derivative, diffusive realization

MOTS-CLÉS : modèle visqueux non local, demi-dérivée de Riemann-Liouville, réalisation diffusive

1. Introduction

1.1. State of the art.

The modeling and the mathematical analysis of viscosity in water wave propagation are challenging issues. In the last decade, P. Liu and T. Orfila [8], and D. Dutykh and F. Dias [6] have independently derived viscous asymptotic models for transient long-wave propagation on viscous shallow water. These effects appear as nonlocal terms in the form of convolution integrals. A one-dimensional nonlinear system is presented in [5].
In their recent work [4], M. Chen et al. investigated theoretically and numerically the decay rate for solutions to the following water wave model with a nonlocal viscous dispersive term as follows

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds + uu_x = \alpha u_{xx}, \tag{1}$$

where $\frac{1}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds$ represents the Caputo half-derivative in time. Here u is the horizontal velocity of the fluid, $-\alpha u_{xx}$ is the usual diffusion, βu_{xxx} is the geometric dispersion and $\frac{1}{\sqrt{\pi}} \int_0^t \frac{u_t(s)}{\sqrt{t-s}} ds$ stands for the nonlocal diffusive-dispersive term. We denote as β, ν and α the parameters dedicated to balance or unbalance the effects of viscosity and dispersion against nonlinear effects. Particularly, the authors in [4] consider (1) with $\beta = 0$ supplemented with the initial condition $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. They proved that if $\|u_0\|_{L^1(\mathbb{R})}$ is small enough, then there exists a unique global solution $u \in C(\mathbb{R}_+; L^2_x(\mathbb{R})) \cap C^1(\mathbb{R}_+; H_x^{-2}(\mathbb{R}))$. In addition, u satisfies

$$t^{1/4} \|u(t,\cdot)\|_{L^2_x(\mathbb{R})} + t^{1/2} \|u(t,\cdot)\|_{L^\infty_x(\mathbb{R})} < C(u_0). \tag{2}$$

In order to study the effects of the nonlocal term on the existence and on the decay rate of the solutions, the second author considered in her recent work [10] a derived model from (1) where the fractional term is described by the Riemann-Liouville half derivative instead of that of Caputo, namely

$$u_t + u_x + \beta u_{xxx} + \frac{\sqrt{\nu}}{\sqrt{\pi}} \frac{\partial}{\partial t} \int_0^t \frac{u(s)}{\sqrt{t-s}} ds + uu_x = \alpha u_{xx}. \tag{3}$$

She proved the local and global existence of solutions to problem (3) when $\beta = 0$ using a fixed point theorem. Then she studied theoretically the decay rate of the solutions in this case. Precisely, she stated the following theorem

Theorem 1.1 (I. Manoubi, 2014) *Let $u_0 \in L^2(\mathbb{R})$, then there exists a unique local solution $u \in C([0,T); L^2_x(\mathbb{R}))$ of (3).*
Moreover for $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, there exists a positive constant $C_0 > 0$ that depends on u_0 such that if $\|u_0\|_{L^1(\mathbb{R})}$ is small enough, there exists a unique global solution $u \in C(\mathbb{R}_+; L^2_x(\mathbb{R})) \cap C^{1/2}(\mathbb{R}_+; H_x^{-2}(\mathbb{R}))$ of (3) given by

$$u(t,x) = [K_{RL}(t,\cdot) \star u_0](x) - N \circledast u^2(t,x), \tag{4}$$

where K_{RL} and N are given by

$$K_{RL}(t,x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} e^{-x^-} (1 - \frac{1}{2} \int_0^{+\infty} e^{-\frac{\mu^2}{4t} - \frac{\mu|x|}{2t} - \frac{\mu}{2}} d\mu),$$

and

$$N(t,x) = \frac{1}{4\sqrt{\pi t}}\partial_x\left(e^{-\frac{x^2}{4t}}e^{-x^-}\left(1-\frac{1}{2}\int_0^{+\infty}e^{-\frac{\mu^2}{4t}-\frac{\mu|x|}{2}-\frac{\mu}{2}}d\mu\right)\right).$$

with $x^- = \frac{|x|-x}{2} = max(-x,0)$, $\star$ *represents the usual convolution product and* $\circledast$ *is the time-space convolution product defined by*

$$v \circledast w(t,x) = \int_0^t\int_{\mathbb{R}} v(t-s,x-y)w(s,y)dsdy.$$

whenever the integrals make sense. In addition, we have the following estimate

$$\max(t^{1/4},t^{3/4})\|u(t,\cdot)\|_{L^2_x(\mathbb{R})} + \max(t^{1/2},t)\|u(t,\cdot)\|_{L^\infty_x(\mathbb{R})} \leq C_0. \quad (5)$$

The proof of this theorem is presented in [10].
However, all these results are performed assuming a smallness condition on the initial data. In order to remove this smallness condition and to investigate the model (3) for a large class of initial data, we introduce here the concept of diffusive realizations for the half-order derivative. This approach was initially developed by Montseny [12], Montseny et al. [14, 15] and Staffans [16]. Diffusive realization make possible to represent nonlocal in time operators, and more generally *causal* pseudo-differential operators, in a state space model formulation where the state belongs to an appropriate Hilbert space. Different applications of this approach can be found in [1, 7, 11, 13].
In this article, we assume that the effects of the geometric dispersion in (3) is less important than the viscosity effects (i.e we take $\beta = 0$ in (3)) and we assume that the other constants are normalized. Thus, our model is reduced as follows

$$u_t + u_x + \frac{1}{\sqrt{\pi}}\frac{\partial}{\partial t}\int_0^t \frac{u(s)}{\sqrt{t-s}}ds + uu_x = u_{xx}. \quad (6)$$

We prove the well posedness of the model (6) for all initial data $u_0 \in H^1(\mathbb{R})$ using the diffusive realization. To this end, we complete the introduction as follows. We first introduce the diffusive formulation of the half-order Riemann-Liouville derivative. Then, we deduce the mathematical model that derives from (6) using the diffusive approach. Finally, we present the main results of this article.
We note that one can consider the general case of the Caputo or Riemann-Liouville fractional derivative of order α where $0 < \alpha < 1$. Comparing the effects of these nonlocal terms with our results is a challenging issue and it may be the subject of a future work. However, choosing another definition of fractional derivative like Atangana-Baleanu derivative or Caputo-Fabrizio derivative in equation (1) must be justified.

1.2. Diffusive formulation of the model

In the literature, there are several diffusive realizations of the Riemann-Liouville half-order derivative. We recall in the following some of these formulations.
First, the diagonal form of the diffusive realization of $D^{1/2}u(t)$ which will be used in the remaining of this article is given for all $t > 0$ by

$$\begin{cases} \partial_t\psi(t,\sigma) = -\sigma\psi(t,\sigma) + u(t), \quad \psi(0,\sigma) = 0, \quad \sigma \in \mathbb{R}^+ \\ D^{1/2}u(t) = \displaystyle\int_0^{+\infty}\frac{1}{\pi\sqrt{\sigma}}\partial_t\psi(t,\sigma)d\sigma. \end{cases} \quad (7)$$

Second, the PDE-form of the diffusive realization of $D^{1/2}u(t)$ is given for all $t > 0$ by

$$\begin{cases} \partial_t \Phi(t,y) = \Phi_{yy}(t,y) + u(t) \otimes \delta_{y=0}, \ \ \Phi(0,y) = 0, \ \ y \in \mathbb{R}, \\ D^{1/2}u(t) = 2 < \delta_{y=0}, \partial_t \Phi(t,y) >_{\mathcal{D}',\mathcal{D}} = 2\dfrac{d}{dt}\Phi(t,0). \end{cases} \tag{8}$$

where $\delta_{y=0}$ is the Dirac delta function at $y = 0$ and $u(t) \otimes \delta_{y=0}$ is the tensorial product in the distributions sense of the applications $t \mapsto u(t)$ and $y \mapsto \delta_{y=0}$.
Finally, another form of the diffusive realization of $D^{1/2}u(t)$ is given for all $t > 0$ by

$$\begin{cases} \partial_t \phi(t,\sigma) = -\sigma^2 \phi(t,\sigma) + \dfrac{2}{\pi}u(t), \ \ \phi(0,\sigma) = 0, \ \ \sigma \geq 0, \\ D^{1/2}u(t) = \displaystyle\int_0^{+\infty} (\frac{2}{\pi}u(t) - \sigma^2 \phi(t,\sigma)) d\sigma. \end{cases} \tag{9}$$

We note that the author has used the diffusive realizations (8) and (9) in her PhD Thesis to study mathematically and numerically the integro-differential equation (3) when $\beta = 0$, $\nu = \alpha = 1$. For more details, we refer the readers to [9]. In the following, we describe the mathematical framework. Thanks to the diffusive realization (7), the problem (6) is written as follows

$$\begin{cases} u_t(t,x) + u_x(t,x) + \displaystyle\int_0^{+\infty} (u(t,x) - \sigma\psi(t,x,\sigma)) \frac{d\sigma}{\pi\sqrt{\sigma}} \\ \qquad + u(t,x)u_x(t,x) = u_{xx}(t,x), \ \ t > 0, \ x \in \mathbb{R}, \\ \psi_t(t,x,\sigma) = -\sigma\psi(t,x,\sigma) + u(t,x), \ \ t > 0, \ x \in \mathbb{R}, \ \sigma \geq 0, \\ u(0,x) = u_0(x), \ \ x \in \mathbb{R}, \\ \psi(0,x,\sigma) = 0, \ \ x \in \mathbb{R}, \sigma \geq 0. \end{cases} \tag{10}$$

Then, We rewrite the system (10) as a first-order semi-linear differential equation as follows

$$\begin{cases} X_t + \mathcal{A}X = F(X), \\ X(0) = X_0, \end{cases} \tag{11}$$

where $X = (u,\psi)^T$, $X_0 = (u_0, 0)^T$ and

$$\mathcal{A}X = \begin{pmatrix} \displaystyle\int_0^{+\infty} (u - \sigma\psi) \frac{d\sigma}{\pi\sqrt{\sigma}} - u_{xx} \\ -u + \sigma\psi \end{pmatrix},$$

$$F(X) = \begin{pmatrix} -u_x - uu_x \\ 0 \end{pmatrix}. \tag{12}$$

1.3. Main results.

We introduce our functional space . First, we define the positive measure dN on $\mathbb{R}_+$ by

$$dN(\sigma) = \frac{d\sigma}{\pi\sqrt{\sigma}}.$$

Hence, dN satisfies

$$C_N = \int_0^{+\infty} \frac{dN(\sigma)}{1+\sigma} = 1. \tag{13}$$

Then, we define the spaces

$$\begin{aligned} H_N &= L^2(\mathbb{R}_+, dN), \\ \widetilde{H}_N &= L^2(\mathbb{R}_+, \sigma dN), \\ V &= L^2(\mathbb{R}_+, (1+\sigma)dN). \end{aligned}$$

We suppose that (11) has a regular solution. The following result holds.

Proposition 1.2 *The energy function associated to* (11)

$$\mathcal{E}(t) = \frac{1}{2}\|u(t)\|^2_{L^2_x} + \frac{1}{2}\|\psi(t)\|^2_{L^2(\mathbb{R},\widetilde{H}_N)}, \tag{14}$$

satisfies the following energetic equilibrium

$$\frac{1}{2}\frac{d}{dt}\mathcal{E}(t) = -\int_{\mathbb{R}} \|u(t,x) - \sigma\psi(t,x,\sigma)\|^2_{H_N} dx - \int_{\mathbb{R}} |u_x(t,x)|^2 dx. \tag{15}$$

The natural energy space of the solution X is

$$\mathcal{H} = L^2(\mathbb{R}) \times L^2(\mathbb{R}, \widetilde{H}_N),$$

endowed with the scalar product $(\cdot,\cdot)_{\mathcal{H}}$ defined for all $X = (u,\psi)^T$ and $Y = (v,\chi)^T$ in $\mathcal{H}$ by

$$(X,Y)_{\mathcal{H}} = (u,v)_{L^2(\mathbb{R})} + \int_{\mathbb{R}} (\psi,\chi)_{\widetilde{H}_N} dx.$$

Moreover, we define the following Hilbert space

$$\mathcal{V} = H^2(\mathbb{R}) \times L^2(\mathbb{R}, \widetilde{H}_N).$$

We state the main result of this article.

Theorem 1.3 *For all $u_0 \in H^1(\mathbb{R})$, there exists a unique global solution $X \in C([0,+\infty[, D(\mathcal{A}^{1/2}))$ of* (11) *such that $X_0 = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$ and*

$$X(t) = \Phi(X)(t). \tag{16}$$

2. Proof of Theorem 1.3.

2.1. The linear problem

We first consider the following linear problem associated to (11)

$$\begin{cases} X_t + \mathcal{A}X = 0 \ \ \forall t > 0, \\ X(0) = X_0, \end{cases} \tag{17}$$

where $X = \begin{pmatrix} u \\ \psi \end{pmatrix}$, $X_0 = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}$ and

$$\mathcal{A}X = \begin{pmatrix} \displaystyle\int_0^{+\infty} (u - \sigma\psi)\frac{d\sigma}{\pi\sqrt{\sigma}} - u_{xx} \\ -u + \sigma\psi \end{pmatrix}, \tag{18}$$

We can establish the following properties of the operator $\mathcal{A}$.

Proposition 2.1 *The domain $D(\mathcal{A})$ of the operator $\mathcal{A}$ in $\mathcal{H}$ is given by*

$$D(\mathcal{A}) = \{(u, \psi) \in \mathcal{V}; u - \sigma\psi \in L^2(\mathbb{R}, V)\}.$$

We define the norm of $X \in D(\mathcal{A})$ by

$$\|X\|_{D(\mathcal{A})} = \left(\|X\|_{\mathcal{H}}^2 + \|\mathcal{A}X\|_{\mathcal{H}}^2\right)^{1/2}.$$

Moreover $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is well-defined and bounded on $D(\mathcal{A})$.

Lemma 2.2 *The domain $D(\mathcal{A}^{1/2})$ of the operator $\mathcal{A}^{1/2}$ in $\mathcal{H}$ is given by*

$$D(\mathcal{A}^{1/2}) = \{(u, \psi) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}, \widetilde{H}_N) \text{ and } u - \sigma\psi \in L^2(\mathbb{R}, H_N)\}.$$

equipped by the norm

$$\|X\|_{D(\mathcal{A}^{1/2})} = \left(\int_{\mathbb{R}} \|u - \sigma\psi\|_{H_N}^2 dx + \|u_x\|_{L^2(\mathbb{R})}^2\right)^{1/2}.$$

Proposition 2.3 *The operator $\mathcal{A}$ is maximal monotone and symmetric. Thus $\mathcal{A}$ is auto-adjoint.*

In the following, we state results based on the Hille-Yosida Theorem [2, 3].

Proposition 2.4 (Hille-Yosida) *For all $X_0 \in D(\mathcal{A})$, there exists a unique solution*

$$X \in C^1(]0, +\infty[, \mathcal{H}) \bigcap C([0, +\infty[, D(\mathcal{A}))$$

of (17). *Moreover, formally we have*

$$X(t) = e^{-t\mathcal{A}} X_0.$$

Proposition 2.5 *For all $X_0 \in \mathcal{H}$, there exists a unique solution*

$$X \in C([0, +\infty[, \mathcal{H}) \bigcap C^1(]0, +\infty[, \mathcal{H}) \bigcap C(]0, +\infty[, D(\mathcal{A}))$$

of (17).

Proposition 2.6 *For all $X_0 \in D(\mathcal{A}^{1/2})$, equation* (17) *has a unique solution*

$$X \in C([0, +\infty[, D(\mathcal{A}^{1/2})).$$

Proof. Let $X_0 \in D(\mathcal{A}^{1/2})$. We consider the following problem

$$\begin{cases} \mathcal{H}_1 = D(\mathcal{A}^{1/2}), \\ \mathcal{A}_1 X = \mathcal{A}X \quad \text{pour} \quad X \in \mathcal{H}_1, \\ X_t + \mathcal{A}_1 X = 0 \ \ \forall t > 0, \\ X(0) = X_0 \in \mathcal{H}_1. \end{cases} \tag{19}$$

Then $\mathcal{A}_1$ is unbounded operator and $D(\mathcal{A}_1)$ is its domain in $\mathcal{H}_1$. By construction, $\mathcal{A}_1$ is a self-adjoint operator. Moreover,

$$\begin{aligned} D(\mathcal{A}_1) =& \{X \in \mathcal{H}_1; \mathcal{A}_1 X \in \mathcal{H}_1\} \\ =& \{X \in D(\mathcal{A}^{1/2}); \mathcal{A}X \in D(\mathcal{A}^{1/2})\} \\ =& \{X \in D(\mathcal{A}); (\mathcal{A}^2 X, \mathcal{A}X) < \infty\} = D(\mathcal{A}^{3/2}). \end{aligned}$$

In addition, $\mathcal{A}_1$ is a maximal and monotone operator. In fact,

$$(\mathcal{A}_1 X, X)_{\mathcal{H}_1} = (\mathcal{A}X, X)_{D(\mathcal{A}^{1/2})} = (\mathcal{A}\mathcal{A}X, X)_{\mathcal{H}}.$$

Since $\mathcal{A}$ is self-adjoint then

$$(\mathcal{A}_1 X, X)_{\mathcal{H}_1} = (\mathcal{A}X, \mathcal{A}X)_{\mathcal{H}} = \|\mathcal{A}X\|^2_{\mathcal{H}} \geq 0.$$

We deduce that $\mathcal{A}_1$ is monotone. Now, we establish that $\mathcal{A}_1$ is maximal. Let $Y \in \mathcal{H}_1 = D(\mathcal{A}^{1/2})$ and we establish that there exists $X \in D(\mathcal{A}_1)$ such that $(I + \mathcal{A}_1)X = Y$. Since $\mathcal{H}_1 \subset \mathcal{H}$ then there exists $X \in D(\mathcal{A})$ such that

$$(I + \mathcal{A})X = X + \mathcal{A}X = Y.$$

In particular, since $D(\mathcal{A})) \subset D(\mathcal{A}^{1/2})$, then

$$X \in D(\mathcal{A}^{1/2}) \text{ et } Y \in D(\mathcal{A}^{1/2}),$$

This implies that

$$X \in D(\mathcal{A}^{1/2}) \text{ et } \mathcal{A}X \in D(\mathcal{A}^{1/2}).$$

We conclude that $X \in D(\mathcal{A}^{3/2}) = D(\mathcal{A}_1)$ and verifies $(I + \mathcal{A})X = (I + \mathcal{A}_1)X = Y$. Hence, using Hille-Yosida Theorem, we conclude that there exists a unique solution of (19)

$$X(t) = e^{-t\mathcal{A}_1} X_0 \in C^0([0, +\infty[, \mathcal{H}_1).$$

Moreover, $D(\mathcal{A}^{1/2}) = \mathcal{H}_1 \subset \mathcal{H}$ and using the uniqueness of the solution of (17), we deduce that

$$X(t) = e^{-t\mathcal{A}} X_0 \in C^0([0, +\infty[, D(\mathcal{A}^{1/2})).$$

■

We have the following uniform estimates.

Proposition 2.7 *First,*

$$\forall X_0 \in \mathcal{H}, \forall t > 0, \;\; \|e^{-t\mathcal{A}}X_0\|_{\mathcal{H}} \leq \|X_0\|_{\mathcal{H}}. \tag{20}$$

Second,

$$\forall X_0 \in D(\mathcal{A}^{1/2}), \forall t > 0, \;\; \|e^{-t\mathcal{A}}X_0\|_{D(\mathcal{A}^{1/2})} \leq \|X_0\|_{D(\mathcal{A}^{1/2})}. \tag{21}$$

Finally,

$$\exists C > 0, \forall X_0 \in \mathcal{H}, \forall t > 0, \;\; \|e^{-t\mathcal{A}}X_0\|_{D(\mathcal{A}^{1/2})} \leq \frac{C}{\sqrt{t}}\|X_0\|_{\mathcal{H}}. \tag{22}$$

2.2. Resolution in $H^1(\mathbb{R})$.

In this subsection, we focus on the problem (11). Formally, if X is a solution of (11) then X satisfies the Duhamel form as follows

$$X(t) = e^{-t\mathcal{A}}X_0 + \int_0^t e^{-(t-s)\mathcal{A}}F(X(s))ds, \tag{23}$$

Hence X is considered as a fixed point of the functional Φ defined by (23) as

$$\Phi(X)(t,x) = e^{-t\mathcal{A}}X_0 + \int_0^t e^{-(t-s)\mathcal{A}}F(X(s))ds. \tag{24}$$

In the sequel, we establish Theorem 1.3. To this end, we start by proving the local existence of the solution of (16) using the fixed point Theorem.
Local existence. First we have the following result.

Proposition 2.8 *The function $F : D(\mathcal{A}^{1/2}) \subset \mathcal{H} \to \mathcal{H}$, given by* (12)*, is locally lipschitz continuous on $D(\mathcal{A}^{1/2})$. Moreover, for all $X, Y \in D(\mathcal{A}^{1/2})$ we have*

$$\|F(X) - F(Y)\|_{\mathcal{H}} \leq (\|X\|_{D(\mathcal{A}^{1/2})} + \|Y\|_{D(\mathcal{A}^{1/2})} + 1)\|X - Y\|_{D(\mathcal{A}^{1/2})}.$$

Corollory 2.9 *Since $F(0) = 0$, we deduce that*

$$\forall X \in D(\mathcal{A}^{1/2}), \; \|F(X)\|_{\mathcal{H}} \leq \|X\|^2_{D(\mathcal{A}^{1/2})} + \|X\|_{D(\mathcal{A}^{1/2})}. \tag{25}$$

Let $T > 0$ and set

$$E_T = C([0,T], D(\mathcal{A}^{1/2})),$$

E_T is a Banach space when endowed with the norm

$$\|X\|_{E_T} := \sup_{t\in[0,T]} \|X(t)\|_{D(\mathcal{A}^{1/2})}.$$

In the following, we state some properties satisfied by the functional Φ on E_T with $X_0 \in D(\mathcal{A}^{1/2})$). Let $X \in E_T$, we have

$$\begin{aligned}\|\Phi(X)(t)\|_{D(\mathcal{A}^{1/2})} &\leq \|e^{-t\mathcal{A}}X_0\|_{D(\mathcal{A}^{1/2})} + \int_0^t \|e^{-(t-s)\mathcal{A}}F(X(s))\|_{D(\mathcal{A}^{1/2})}ds \\ &\leq \|X_0\|_{D(\mathcal{A}^{1/2})} + \int_0^t \frac{c}{\sqrt{t-s}}\|F(X(s))\|_{\mathcal{H}}ds.\end{aligned}$$

Thanks to (25), we obtain

$$\|\Phi(X)(t)\|_{D(\mathcal{A}^{1/2})} \leq \|X_0\|_{D(\mathcal{A}^{1/2})} + \int_0^t \frac{c}{\sqrt{t-s}}(\|X(t)\|^2_{D(\mathcal{A}^{1/2})} + \|X(t)\|_{D(\mathcal{A}^{1/2})})ds.$$

Hence, for all $t \in [0,T]$ we have

$$\|\Phi(X)(t)\|_{D(\mathcal{A}^{1/2})} \leq \|X_0\|_{D(\mathcal{A}^{1/2})} + C_1\sqrt{T}(\|X\|^2_{E_T} + \|X\|_{E_T}). \tag{26}$$

Moreover, let X and $Y \in E_T$ then

$$\begin{aligned}
\|\Phi(X)(t) - \Phi(Y)(t)\|_{D(\mathcal{A}^{1/2})} &= \| \int_0^t e^{-(t-s)\mathcal{A}}F(X(s))ds - \int_0^t e^{-(t-s)\mathcal{A}}F(Y(s))ds\|_{D(\mathcal{A}^{1/2})} \\
&\leq \int_0^t \|e^{-(t-s)\mathcal{A}}(F(X(s)) - F(Y(s)))\|_{D(\mathcal{A}^{1/2})}ds \\
&\leq \int_0^t \frac{c}{\sqrt{t-s}}\|F(X(s)) - F(Y(s))\|_{\mathcal{H}}ds \\
&\leq \int_0^t \frac{c(K)}{\sqrt{t-s}}\|X(s) - Y(s)\|_{D(\mathcal{A}^{1/2})}ds \\
&\leq c(K)\sqrt{t}\|X - Y\|_{E_T}.
\end{aligned}$$

Here K is the constant of Lipschitz of F on the ball B. Hence, for all $t \in [0,T]$

$$\|\Phi(X)(t) - \Phi(Y)(t)\|_{D(\mathcal{A}^{1/2})} \leq C_2\sqrt{T}\|X - Y\|_{E_T}. \tag{27}$$

Also, we show that if $X_0 \in D(\mathcal{A}^{1/2}))$ then Φ is well defined. Next we define a set B invariant under the action of Φ. Therefore we take $R = 2\|X_0\|_{D(\mathcal{A}^{1/2})}$. Let $B(0,R)$ the closed ball in E_T of radius R centered at the origin. Thanks to (26) and (27), we get

$$\forall X \in B, \ \ \|\Phi(X)(t)\|_{D(\mathcal{A}^{1/2})} \leq \frac{R}{2} + C_1\sqrt{T}(R^2 + R),$$
$$\forall X, Y \in B, \ \ \|\Phi(X)(t) - \Phi(Y)(t)\|_{D(\mathcal{A}^{1/2})} \leq C_2\sqrt{T}\|X - Y\|_{E_T}.$$

Finally, we choose T small such that $\max(C_1R, C_2)\sqrt{T} \leq \frac{1}{2}$. Hence, with this choice, we get $\Phi(B) \subset B$ and thus the map Φ is a contraction on B. Using the fixed point Theorem, we deduce that there exists a unique fixed point X of the functional Φ on B. Moreover, $X \in C([0,T], D(\mathcal{A}^{1/2}))$.

In the following, we establish the global existence of the solution of (16).

Global existence. We take the scalar product in $\mathcal{H}$ of (11) with X, we get

$$(X_t, X)_{\mathcal{H}} + (\mathcal{A}X, X)_{\mathcal{H}} = (F(X), X)_{\mathcal{H}}. \tag{28}$$

We observe that

$$(F(X), X)_{\mathcal{H}} = \left(\begin{pmatrix} -u_x - uu_x \\ 0 \end{pmatrix}, \begin{pmatrix} u \\ \psi \end{pmatrix}\right)_{\mathcal{H}} = \int_{\mathbb{R}} (-u_x - uu_x)udx = 0.$$

Hence (28) is written as

$$\frac{1}{2}\frac{d}{dt}\|X(t)\|^2_{\mathcal{H}} + \|X(t)\|^2_{D(\mathcal{A}^{1/2})} = 0.$$

We deduce that there exists a constant $C > 0$ such that

$$\forall t > 0, \int_0^t \|X(s)\|^2_{D(\mathcal{A}^{1/2})} ds \leq C\|X_0\|^2_{\mathcal{H}} = C\|u_0\|^2_{L^2(\mathbb{R})}. \tag{29}$$

Moreover, we take the scalar product in $\mathcal{H}$ of (11) with $\mathcal{A}X$. Since $\mathcal{A}$ is self-adjoint, we get

$$\frac{1}{2}\frac{d}{dt}\|X\|^2_{D(\mathcal{A}^{1/2})} + \|\mathcal{A}X\|^2_{\mathcal{H}} = (F(X), \mathcal{A}X)_{\mathcal{H}}. \tag{30}$$

Moreover, using Cauchy-Schwarz inequality and the estimation (25), we get

$$(F(X), \mathcal{A}X)_{\mathcal{H}} \leq \|F(X)\|_{\mathcal{H}}\|\mathcal{A}X\|_{\mathcal{H}} \leq (\|X\|^2_{D(\mathcal{A}^{1/2})} + \|X\|_{D(\mathcal{A}^{1/2})})\|\mathcal{A}X\|_{\mathcal{H}}.$$

Using Young inequality, we obtain

$$\begin{aligned}(F(X), \mathcal{A}X)_{\mathcal{H}} &\leq c(\|X\|^2_{D(\mathcal{A}^{1/2})} + \|X\|_{D(\mathcal{A}^{1/2})})^2 + \frac{1}{2}\|\mathcal{A}X\|^2_{\mathcal{H}} \\ &\leq c\|X\|^4_{D(\mathcal{A}^{1/2})} + \|X\|^2_{D(\mathcal{A}^{1/2})} + \frac{1}{2}\|\mathcal{A}X\|^2_{\mathcal{H}}.\end{aligned}$$

We deduce using (30) that for all $t \in [0, T]$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|X(t)\|^2_{D(\mathcal{A}^{1/2})} &\leq c\|X(t)\|^4_{D(\mathcal{A}^{1/2})} + \|X(t)\|^2_{D(\mathcal{A}^{1/2})} \\ &\leq c\|X(t)\|^2_{D(\mathcal{A}^{1/2})}(\|X(t)\|^2_{D(\mathcal{A}^{1/2})} + 1).\end{aligned}$$

Then using Gronwall inequality, we get for all $t \in [0, T]$

$$\begin{aligned}\|X(t)\|^2_{D(\mathcal{A}^{1/2})} &\leq \|X_0\|^2_{D(\mathcal{A}^{1/2})} \exp(c\int_0^t (\|X(s)\|^2_{D(\mathcal{A}^{1/2})} + 1)ds) \\ &\leq \|X_0\|^2_{D(\mathcal{A}^{1/2})} \exp(ct)\exp(c\int_0^t \|X(s)\|^2_{D(\mathcal{A}^{1/2})}ds).\end{aligned}$$

Finally, taking in account the estimation (29), we deduce that for all $t \in [0, T]$

$$\|X(t)\|^2_{D(\mathcal{A}^{1/2})} \leq Ce^{ct}\|X_0\|^2_{D(\mathcal{A}^{1/2})}.$$

Let $T_{max} \in]0, +\infty]$ be the maximal existence time of the solution X of (11). We have

$$\forall t < T_{max}, \ \|X(t)\|^2_{D(\mathcal{A}^{1/2})} \leq Ce^{ct}\|X_0\|^2_{D(\mathcal{A}^{1/2})}. \tag{31}$$

We deduce that $T_{max} = +\infty$.

3. Conclusion

In this article, we investigate theoretically the well-posedness of an asymptotical water wave model with a nonlocal viscous term described by the Riemann-Liouville half derivative. Here we present the half-derivative using a diffusive realization. We proved the existence and the uniqueness of solutions for all initial data $u_0 \in H^1(\mathbb{R})$. A challenging issue is to study theoretically and numerically the decay rate of solutions for this class of initial data. This question will be the subject of a future work.

4. References

[1] J. AUDOUNET, V. GIOVANGIGLI, J. ROQUEJOFFRE, "A threshold phenomenon in the propagation of a point-source initiated flame", *Physica D*, vol. 121, 1998.

[2] H. BREZIS, " Analyse Fonctionnelle : Théorie et Applications ", *Editions Masson, Paris*, 1983.

[3] T. CAZENAVE , A. HARAUX, " An Introduction to Semilinear Evolution Equations ", *Oxford University Press, Oxford*, 1998.

[4] M. CHEN, S. DumontL. DUPAIGNE, O. GOUBET " Decay of solutions to a water wave model with a nonlocal viscous dispersive term ", *Discrete Contin. Dyn. Syst.*, vol. 27, 2010.

[5] D. DUTYKH, " Viscous-potential free-surface flows and long wave modelling ", *Eur. J. Mech. B Fluids*, vol. 28, 2009.

[6] D. DUTYKH , F. DIAS, " Viscous potential free surface flows in a fluid layer of finite depth *C.R.A.S, Série I*, vol. 345, 2007.

[7] T. HELIE, D. MATIGNON, " Diffusive reprentations for the analysis and simulation of flared acoustic pipes with visco-thermal losse ", *Mathematical Models and Methods in Applied Sciences*, vol. 16, 2006.

[8] P. LIU, A. ORFILA " Viscous effects on transient long wave propagation " *J. Fluid Mech.*, vol. 520 , 2004.

[9] I. MANOUBI, " Modèle visqueux asymptotique pour la propagation d'une onde dans un canal " *PhD Thesis*, 2014.

[10] I. MANOUBI, " Theoretical and numerical analysis of the decay rate of solutions to a water wave model with a nonlocal viscous dispersive term with Riemann-Liouville half derivative " *Discrete Contin. Dyn. Syst*, vol. 19, 2014.

[11] D. MATIGNON, C. PRIEUR, " Asymptotic stability of linear conservative systems when coupled with diffusive systems", *ESAIM: Control, Optim. and Calc. of Var.*, vol. 11, 2005.

[12] G. MONTSENY , " Diffusion monodimensionnelle et intégration d'ordre $1/2$ ", *Internal LAAS Report N.* 91232, 1991.

[13] G. MONTSENY, J. AUDOUNET , D. MATIGNON, " Diffusive representation for pseudodifferentially damped nonlinear systems ", *Nonlinear control in the year 2000*, vol. 2, 2000.

[14] G. MONTSENY, J. AUDOUNET , B. MBODGE, " Modèle simple d'amortisseur viscoélastique. Application à une corde vibrante " , *Lecture notes in Control and Information Sciences, Eds. RF.Curtain, A.Bensoussan, JL.Lions-Springer Verlag*, vol. 185 (1993),

[15] G. MONTSENY, J. AUDOUNET , B. MBODGE, " Optimal models of fractional integrators and application to systems with fading memory ", *IEEE International Conference on Systems, Man and Cybernetics, Le Touquet France*, 1993.

[16] O. STAFFANS , " Well-posedness and stabilizability of a viscoelastic equation in energy space ", *Transactions of the American Mathematical Society*, vol. 345, 1994.