Rubrique

Well's location in porous media using topological asymptotic expansion

Wafa Mansouri[1,2], Thouraya Nouri Baranger[2], Hend Ben Ameur[1] and Nejla Tlatli[1]

[1] National Engineering School of Tunis
University of Tunis El Manar
wafa.mansouri@insa-lyon.fr
hbenameur@yahoo.ca
tlatli@topnet.tn
[2] University of Lyon, CNRS, University lyon1
LAMCOS UMR5259, INSA-Lyon
Thouraya.Baranger@univ-lyon1.fr

ABSTRACT. We study the inverse problem of identification of well's location in a porous media via boundary measurements. Our main tool is the topological gradient method applied to a convenient design function.

RÉSUMÉ. Nous étudions le problème inverse d'identification des positions des puits dans un milieu poreux par des mesures sur la frontière. Notre outil principal est la méthode du gradient topologique appliqué à une fonction objectif.

KEYWORDS : Inverse Problem, Topological Gradient, Wells Location.

MOTS-CLÉS : Problème Inverse, Gradient Topologique, Identifications des Puits.

1 Introduction

In hydrogeology, it is very difficult to construct an accurate simulation model for a groundwater system. However, in many real situations, uncertainties can be related to parameters caracterising the aquifer itself, or to external constraints, such as withdrawal rates in wells, drilling and recharge. The knowledge of the aquifer withdrawal rates can represent a largely unknown factor in real problems of groundwater resources modelling. The inverse problem under consideration is to determine the location of wells using boundary measurements. We consider the cases where we have Neumann or Dirichlet condition on boundary of the wells and we use the topological sensitivity method.

The topological sensitivity analysis has been recognized as a promising method to solve topology optimization problems. It consists to derive an asymptotic expansion of a shape functional with respect to the size of a small hole created inside the domain. This method was introduced by Schumacher [12] in the context of compliance minimization. Then, Sokolowski and Zochowski [13] generalized it to more general shape functionals.

To present the basic idea of this method, let us consider a domain Ω in $\mathbb{R}^2$ and a cost functional $j(\Omega) = J(u_\Omega)$ to be minimized, where u_Ω is the solution to a given PDE (model) defined in Ω. For a small parameter $\varepsilon \geq 0$, let $\Omega \backslash \overline{B(x_0, \varepsilon)}$ be the perturbed domain obtained by the creation of a circular hole of radius ε around the point $x_0 \in \Omega$. The topological sensitivity analysis provides an asymptotic expansion of j when ε goes to zero in the form:

$$j(\Omega \backslash \overline{B(x_0, \varepsilon)}) - j(\Omega) = f(\varepsilon) g(x_0) + o(f(\varepsilon)). \quad (1)$$

In this expansion, $f(\varepsilon)$ is a positive function going to zero with ε. The function g is commonly called topological gradient, or topological derivative. It is usually simple to compute and is obtained using the solution of direct and adjoint problems defined on the initial domain. To minimize the criterion j, one has to create holes at some points x where $g(x)$ is negative.
The topological derivative has been obtained for various problems, arbitrary shaped holes and a large class of shape functionals [5, 10].

This work is outlined as follows: Section 2 is devoted to the model setting; section 3 is devoted to the formulation of the inverse problem and the introduction of the topological asymptotic analysis in the case of a well with a Dirichlet or Neumann condition on its boundary. In section 4 we illustrate the efficiency of the proposed method by several numerical experiments then we conclude.

2 The model setting

Let Ω be a domain of $\mathbb{R}^2$ and $\Gamma = \partial\Omega$. We assume that the wells are well separated and have a circular form $\mathcal{O}_{x_k,\varepsilon} = x_k + \varepsilon\mathcal{O}^k$, $1 \leq k \leq m$, where ε is the common diameter and $\mathcal{O}^k \subset \mathbb{R}^2$ are bounded and smooth domains containing the origin. The points $x_k \in \Omega$, $1 \leq k \leq m$, determine the location of the wells (Figure 1). These are the unknowns of our inverse problem.

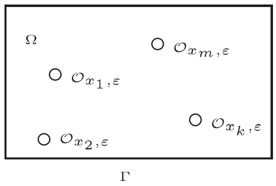


Figure 1: Domain containing m wells having the same radius.

For simplicity, we assume that Ω is a homogeneous geological zone. Following [4], the forward two-dimensional problem of groundwater flow in isotropic and homogeneous medium, with wells $\mathcal{O}_\varepsilon = \bigcup_{k=1}^{m} \mathcal{O}_{x_k,\varepsilon}$ in the domain Ω, can be formulated as follows:

$$\begin{cases} -div(T\nabla u_\varepsilon) &= 0 & \text{in } \Omega\backslash\overline{\mathcal{O}_\varepsilon} \\ u_\varepsilon &= H & \text{on } \Gamma \\ T\nabla u_\varepsilon.n &= 0 \ \text{(or } u_\varepsilon = 0) & \text{on } \Sigma_\varepsilon \end{cases} \tag{2}$$

Where $\Sigma_\varepsilon = \partial\mathcal{O}_\varepsilon$ is the wells boundary; T is the transmissivity and u_ε is the pieozometric head . Let $\Omega_\varepsilon = \Omega\backslash\overline{\mathcal{O}_\varepsilon}$.

3 The inverse problem and the misfit function

We consider the inverse problem of determining well's location from overspecified boundary data on Γ. These data correspond to both Neumann and Dirichlet conditions. We split these data in such a way to build two well posed problems:

- The first problem uses a Neumann condition on Γ,

$$\begin{cases} -div(T\nabla u_\varepsilon^N) &= 0 & \text{in } \Omega_\varepsilon \\ T\nabla u_\varepsilon^N.n &= \Phi & \text{on } \Gamma \\ T\nabla u_\varepsilon^N.n &= 0 \ \text{(or } u_\varepsilon^N = 0) & \text{on } \Sigma_\varepsilon \end{cases} \tag{3}$$

- The second problem uses a Dirichlet condition on Γ,

$$\begin{cases} -div(T\nabla u_\varepsilon^D) &= 0 & \text{in } \Omega_\varepsilon \\ u_\varepsilon^D &= H & \text{on } \Gamma \\ T\nabla u_\varepsilon^D.n &= 0 \ \text{(or } u_\varepsilon^D = 0) & \text{on } \Sigma_\varepsilon \end{cases} \tag{4}$$

We define a misfit function:

$$J(u_\varepsilon^D, u_\varepsilon^N) = \|u_\varepsilon^N - u_\varepsilon^D\|_{L^2(\Omega_\varepsilon)}^2 \tag{5}$$

One can remark that if Σ_ε coincides with the actual well boundary Σ_ε^* then the misfit between the solutions vanishes $u_\varepsilon^N = u_\varepsilon^D$.

Our identification problem can be formulated as a topological optimization problem as follows: given a flow ϕ and the measured H, find the optimal location of wells $\mathcal{O}_\varepsilon$ inside the domain Ω minimizing the shape function j

$$(\mathcal{P}_{min}) \quad \min_{\mathcal{O}_\varepsilon \subset \Omega} j(\varepsilon)$$

where

$$j(\varepsilon) = J(u_\varepsilon^D, u_\varepsilon^N). \tag{6}$$

The solution of this inverse problem depends on the boundary condition on the well's boundary taken in (4) and (3).

3.1 Case 1: the topological gradient considering Neumann condition on Σ_ε

The aim of this section is to derive a topological asymptotic expansion for equation (7) with Neumann condition on Σ_ε:

$$\begin{cases} -div(T\nabla u_\varepsilon) & = 0 & \text{in } \Omega_\varepsilon \\ T\nabla u_\varepsilon.n & = \Phi & \text{on } \Gamma \\ u_\varepsilon & = H & \text{on } \Gamma \\ T\nabla u_\varepsilon.n & = 0 & \text{on } \Sigma_\varepsilon \end{cases} \tag{7}$$

A topological sensitivity analysis using Neumann boundary condition has already been obtained for the elasticity equations in [5], for Laplace equation in [2] and for Maxwell equations in [9].

Inspired by the master thesis work [8], the topological gradient method provides an asymptotic expansion of a function j defined in (6) of the form:

$$\begin{aligned} j(\varepsilon) - j(0) &= -2\pi T\varepsilon^2[\nabla u_0^D(z)\nabla v_0^D(z) + \nabla u_0^N(z).\nabla v_0^N(z) \\ &+ \frac{1}{2}|u_0^N(z) - u_0^D(z)|^2] + o(\varepsilon^2), \end{aligned}$$

In this case the topological gradient is defined by:

$$g(z) = -2\pi T[\nabla u_0^D(z)\nabla v_0^D(z) + \nabla u_0^N(z).\nabla v_0^N(z) + \frac{1}{2}|u_0^N(z) - u_0^D(z)|^2]$$

where u_0^N and u_0^D are respectively the solution of the problems (3) and (4) with $\varepsilon = 0$ (in the domain without wells). Then v_0^N and v_0^D are respectively the solution of the adjoint problems associated to the problems (3) and (4) in the domain without wells.

3.2 Case 2: the topological gradient considering Dirichlet condition on Σ_ε

Instead to the case 1, our goal here is to present a topological asymptotic expansion for equation (8) with Dirichlet condition on Σ_ε:

$$\begin{cases} -div(T\nabla u_\varepsilon) & = 0 & \text{in } \Omega_\varepsilon \\ T\nabla u_\varepsilon.n & = \Phi & \text{on } \Gamma \\ u_\varepsilon & = H & \text{on } \Gamma \\ u_\varepsilon & = 0 & \text{on } \Sigma_\varepsilon \end{cases} \tag{8}$$

Topological sensitivity analysis with Dirichlet boundary condition on the boundary of the hole $\mathcal{O}_\varepsilon$ was considered in [6] for Stokes equations, in [7] for quasi-Stokes equations and in [1] for Navier-Stokes equations.

In this section, we derive a topological asymptotic expansion for function j. It consists in studying the variation of j with respect to the presence of a small wells $\mathcal{O}_\varepsilon$ with a

Dirichlet boundary condition on Σ_ε. As mentioned above, we will derive an asymptotic expansion for j on the form:

$$j(\varepsilon) - j(0) = \frac{-2T\pi}{log(\varepsilon)}[u_0^N(z)v_0^N(z) + u_0^D(z)v_0^D(z)] + o(\frac{-1}{log(\varepsilon)}),$$

In this case the topological gradient is defined by:

$$g(z) = -2\pi T[u_0^N(z)v_0^N(z) + u_0^D(z)v_0^D(z)].$$

4 Numerical result

4.1 One-shot reconstruction algorithm

The identification procedure is a one shot algorithm based on the following steps:

– Step 1: solve the direct and adjoint problems,

– Step 2: compute the topological gradient g,

– Step 3: determine the negative local minima of g.

To test the efficiency of the proposed reconstruction process, different cases are studied. Wells are likely to be located at spots where the topological gradient g is most negative. The discretization of the direct problems (4) and (3) for $\varepsilon = 0$ with Dirichlet or Neumann conditions is based on triangular mesh and the finite element method. The numerical simulations are done using a $2D$ version of the software Comsol and Matlab [3].

The numerical tests are performed on a $20\,km \times 10\,km$ rectangular domain, with a homogeneous transmissivity $T = 0.001m^2s^{-1}$.

We define the relative errors as:

$$\tau_z = 100|\frac{\|OP_{ex}\| - \|OP_{id}\|}{\|OP_{ex}\|}|, \tag{9}$$

for position's identification: Where OP the position and O is the origin of the coordinate system. We denote by subscripts *ex* and *id* the exact and identified solutions.

4.2 Effects of mesh size

The aim of these first numerical experiment is to study the influence of mesh size on the results of the algorithm defined on the previous section. We consider a unique well centred at $z_{exact} = (0.3, 0.3)$ and having a radius ε.

Mesh	Mesh size h	Finite elements number	P_{id}	τ_z [%]
1	0.00625	1025	$(0.306, 0.305)$	1.8
2	0.02	736	$(0.31, 0.312)$	3.66
3	0.1	59	$(0.315, 0.315)$	5.16

Table 1: Effects of mesh size for the case of single well located at ($x = 0.3$, $y = 0.3$).

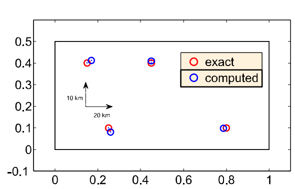
In Table 1, we give a summury of results obtained with different mesh size h. The finer is the mesh the smaller is the error.

4.3 A case of four wells

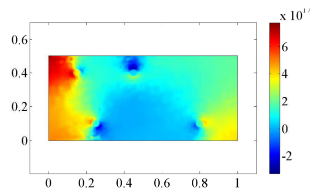
We identify four wells with Neumann (Case 1) and Dirichlet (Case 2) conditions on the boundary of the wells. Computed positions and the relative errors are shown in Table 2. In both cases, errors indicate that we have a good identification. We observe from the Figure 2 that the region where the most negative gradient is located in the vicinity of exact wells' position.

Identified \ Exact	$(0.25, 0.10)$	$(0.45, 0.40)$	$(0.80, 0.10)$	$(0.15, 0.40)$
Case 1	$(0.26, 0.81)$	$(0.45, 0.41)$	$(0.78, 0.09)$	$(0.17, 0.41)$
τ_z [%]	4.29	2	2.82	4.66
Case 2	$(0.24, 0.08)$	$(0.45, 0.41)$	$(0.80, 0.11)$	$(0.16, 0.41)$
τ_z [%]	4.23	3.23	3.68	3.95

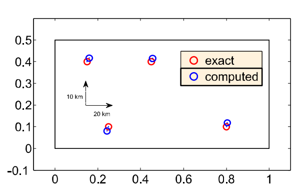
Table 2: Exact and computed wells' locations and corresponding relative errors.



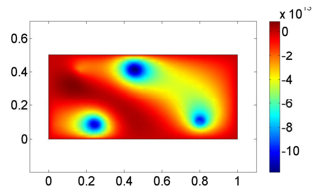
(a) Neumann Condition on well's boundary.



(b) Neumann Condition on well's boundary.



(c) Dirichlet Condition on well's boundary.



(d) Dirichlet Condition on well's boundary.

Figure 2: (a) and (c) represent the exact and estimated positions, (b) and (d) are the topological gradient distribution.

4.4 Sensitivity to the relative position

In the third case, we test the sensitivity to the relative position of two wells. We consider two wells separated by a variable distance d, and we compute the relative error for each distance (see Table 3). One can observe that if the wells are "well separated" (far from each other), the wells' locations are well identified, but when the distance between the wells decreases, the identification process is less accurate.

	d	0.63	0.53	0.4	0.25	0.18	0.15
Case 1	τ_z [%]	2.4	3.5	5.22	9.48	11.37	14.4
Case 2	τ_z [%]	2.1	3.22	4.84	9.16	10.59	14.2

Table 3: Influence of the relative distance between the wells.

4.5 Effect of noisy data

In the hope to try later this algorithm on real experimental data, we are interested in this paragraph to the robustness of the algorithm with respect to noisy data. We consider the case of single well located at point $z_{exact} = (0.4, 0.2)$. Then, we disrupt the over-specified data on Γ by adding relative white noise with different noise level. The results are presented in the Table 4. Notice that for noise level less than 6% the method remains efficient and results are in good agreement with the exact ones, whereas, since the noise level is higher fictitious flaws show up (see Table 4).

	Noise level (%)	4	6	10
Case 1	Location	$(0.429, 0.197)$	$(0.438, 0.191)$	$(0.459, 0.178)$
Error	τ_z [%]	5.83	7.81	12.59
Case 2	Location	$(0.381, 0.192)$	$(0.368, 0.181)$	$(0.354, 0.171)$
Error	τ_z [%]	4.12	7.44	10.87

Table 4: Effects of various noise levels for the case of single well located at ($x = 0.4$, $y = 0.2$).

We consider the case of three well separated located at point $z_i,\ i = 1:3$ having the same radius ε, the coordinates of the wells are $z_1 = (0.15, 0.4)$, $z_2 = (0.8, 0.15)$ and $z_3 = (0.45, 0.4)$. Then, we disrupt the overspecified data on Γ by adding relative white noise with different noise level.

	$Noise\ level$ (%)	4	6	10
Cas I	P^1_{id}	$(0.131, 0.42)$	$(0.123, 0.432)$	$(0.093, 0.476)$
	τ_z [%]	2.98	5.14	13.52
	P^2_{id}	$(0.819, 0.159)$	$(0.829, 0.169)$	$(0.839, 0.175)$
	τ_z [%]	2.5	4	5.29
	P^3_{id}	$(0.455, 0.444)$	$(0.462, 0.44)$	$(0.475, 0.475)$
	τ_z [%]	5.59	5.96	11.57
Cas II	P^1_{id}	$(0.136, 0.412)$	$(0.128, 0.425)$	$(0.102, 0.456)$
	τ_z [%]	1.56	3.89	9.37
	P^2_{id}	$(0.823, 0.165)$	$(0.85, 0.18)$	$(0.89, 0.21)$
	τ_z [%]	3.12	6.74	12.34
	P^3_{id}	$(0.459, 0.401)$	$(0.465, 0.42)$	$(0.472, 0.439)$
	τ_z [%]	1.32	4.07	7.06

Table 5: Effects of various noise levels for three wells.

One can note that for less than 6% of noise the wells are very well located wheras for a noise great than 10% it will be difficult to locate their positions (see Table 5).

5 Conclusion

In this work a new procedure for location of wells from overspecified boundary data based on the minimization of a misfit function type for identifying the positions of the wells. We develop an identification process related to the choice of type of boundary condition on well boundary.

The developed algorithm is fast since it a one shot algorithm. The method seems relevant in all the tested cases: multiplicity of wells with different position and noisy data. However, it is important to note that the present inverse problem is very sensitive to the quantity and quality of *"available"* data. The more over-specified data are available, the better are the recovered boundary data.

References

[1] S. Amstutz, "The topological asymptotic for Navier Stokes equations", *ESAIM, Cont. Optim. Cal. Var*, vol. 3, p. 401-425, 2005.

[2] S. Amstutz, I. Horchani, M. Masmoudi, "Crack detection by the topological gradient methods", *Control and Cybernetics*, vol. 34(1), p. 81-101, 2005.

[3] "COMSOL Multiphysics Modeling Guide", *COPYRIGHT by COMSOL AB*, , 1998-2008.

[4] M. De Marsily, "Cours d'HydrogÃ©ologie", *UniversitÃ© de Paris 5*, 2004.

[5] S. Garreau, Ph. Guillaume, M. Masmoudi, "The topological asymptotic for pde systems: the elasticity case", *SIAM J. Cont. Optim*, vol. 39, p. 1756-1778, 2001.

[6] Ph. Guillaume, K. Sid Idris, "Topological sensitivity and shape optimization for the Stokes equations", *SIAM J. Cont. Optim*, vol. 43, p. 1-31, 2004.

[7] M. Hassine, M. Masmoudi, "The topological sensitivity analysis for the Quasi-Stokes problem", *ESAIM, COCV J*, vol. 10, p. 478-504, 2004.

[8] I. Kallel, "Analyse de sensibilité topologique pour l'opérateur de Laplace anisotrope.", *École Supérieur des Sciences et de Technologie, memoire de master* , 2012.

[9] M. Masmoudi, J. Pommier, B. Samet, "The topological asymptotic expansion for the Maxwell equations and some applications", *J. Inverse Problems*, vol. 21(2), p. 547-564, 2005.

[10] M. Masmoudi, "The topological asymptotic, in: Computational Methods for Control Applications (R. Glowinski, H. Karawada and J. Periaux, eds)", *GAKUTO Internat. Ser. Math. Sci. Appl*, vol. 16, p. 53-72, 2001.

[11] B. Samet, S. Amstutz, M. Masmoudi, "Topological sensitivity analysis", *SIAM J. Control*, vol. 2(5), p. 1523-1544, 2003.

[12] A. Schumacher, "Topologieoptimisierung von Bauteilstrukturen unter Verwendung von Lopchpositionierungkrieterien, Thesis", *Universitat-Gesamthochschule-Siegen*, 1995.

[13] J. SokolowskiA. Zochowski, "On the topological derivative in shape optimization", *SIAM J. Control*, vol. 37, p. 1241-1272, 1999.