Identification of Robin coefficient for Stokes Problem

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RÉSUMÉ. Dans ce travail, on s'intéresse à l'identification d'un coefficient de Robin sur une partie non accessible du bord d'un domaine à partir de données **faiblement surdéterminées** sur la partie accessible. Le modèle est régi par les équations de Stokes. Dans un premier temps, nous utilisons une méthode du type décomposition de domaine pour calculer les composantes inconnues de la vitesse et du tenseur des contraintes, puis nous utilisons ces données pour calculer le coefficient recherché. Nous donnons des tests numériques pour valider la méthode utilisée.

ABSTRACT. In this paper, we deal with the inverse problem of identifying a Robin coefficient on some inaccessible part of a boundary of a domain from the knowledge of **partially overdetermined** data on the accessible part. The underlying PDE's system is the Stokes one. We use a domain decomposition-like method to first recover lacking velocity and stress tensor component. Numerical trials highlights the efficiency of the proposed method.

MOTS-CLÉS : Coefficient de Robin, Conditions aux limites défectueuses, Contrainte de cisaillement, Equations de Stokes, Problème inverse

KEYWORDS : Robin coefficient, Defective boundary condition, Shear stress, Stokes equations, Inverse problem

1. Introduction

Consider an incompressible and homogeneous fluid flow governed by Stokes equations into an open bounded and connected domain $\Omega \subset \mathbb{R}^2$. The boundary $\Gamma = \partial\Omega$ is composed of two parts Γ_c and Γ_i having non-vanishing measure and such that $\Gamma_c \cap \Gamma_i$ is empty. Γ_c is the accessible part, Γ_i is the non accessible one. We formulate our problem as follows :

$$(\mathbb{P}) \left\{ \begin{array}{rlll} -\nu\Delta u + \nabla p & = 0 & \text{in} & \Omega \\ \nabla \cdot u & = 0 & \text{in} & \Omega \\ (\sigma(u)\cdot n)\cdot\tau & = g_c & \text{on} & \Gamma_c \\ u \cdot n & = \Phi_c \cdot n & \text{on} & \Gamma_c \\ \sigma(u)\cdot n + Ru & = 0 & \text{on} & \Gamma_i \end{array} \right. \tag{1}$$

ν is the viscosity of the fluid that we will assume equal to 1, σ denotes the stress tensor $\sigma(u) = \sigma(u,p) = 2\nu D(u) - pI$, where $D(u)$ is the strain tensor defined by : $D(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. n is the outward normal on $\partial\Omega$ and τ is the tangential vector of $\partial\Omega$. R is the Robin coefficient assumed hereafter to be a positive number.
We want to determine the coefficient R from the knowledge of $u.\tau$ on Γ_c.
The method followed here to recover R lies on the recovery of the velocity and the normal stress on the non accessible part Γ_i.
Notice that the boundary condition on the Γ_c is not the Neumann condition regarding the Stokes operator. Thus, this is a non-trivial situation since on the accessible boundary the information on the normal component of the normal stress is unavailable, and only partially overspecified data are given. Nonetheless, this condition is natural, one may refer to [1, 2], for instance, for the description and the background on this boundary condition.
The Cauchy problem is known since Hadamard to be ill posed in the sense that if a solution exists, it does not depend continuously on the data (Φ_c, g_c). Thus, the lack of complete data on the accessible boundary Γ_c may increase the degree of the ill-posedness, and numerically worst behavior is expected.
Our work is motivated first by the study of airway resistance in pneumology which characterizes the patient's ventilation capability and second by the study of the resistivity of a stent which is a medical device used to prevent rupture of aneurysms [3, 4].
The problem of identifying Robin coefficient has been studied by Chaabane and Jaoua [5] for Laplace equations and by Boulakia, Egloffe and Grandmont [6] for Stokes problem where they consider the full overdetermined problem namely the velocity and the hole stress tensor on Γ_c.
In our case the difficulty is increased as long as the overdetermined data are incomplete. Contrary to the case considered in [6], there is no unique continuation results helping us to prove identifiability results. Neverthless, the authors have studied in [7] the problem of recovering the velocity and the stress tensor on the inaccessible part of the boundary from these incomplete data on the accessible part and made a full study which will be of great help for the present work.

2. Recovering lacking data

Giving a compatible data $(\Phi_c, g_c) \in (H^{\frac{1}{2}}(\Gamma_c))^2 \times H^{-\frac{1}{2}}(\Gamma_c)$, that is a data for which a solution (u,p) exists for the problem :

$$(\mathbb{PI}) \begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in} \quad \Omega \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \\ u = \Phi_c & \text{on} \quad \Gamma_c \\ (\sigma(u)\cdot n)\cdot \tau = g_c & \text{on} \quad \Gamma_c \end{cases} \tag{2}$$

we want to determine the velocity Φ_i together with $G_i = \sigma(u_i) \cdot n$ on the non accessible part Γ_i.
Assume that Φ_i and G_i are recovered, we will have therefore the following partially over-determined boundary conditions system :

$$\begin{cases} -\nu\Delta u + \nabla p = 0 & \text{in} \quad \Omega \\ \nabla \cdot u = 0 & \text{in} \quad \Omega \\ u = \Phi_c, \ (\sigma(u)\cdot n)\cdot \tau = g_c & \text{on} \quad \Gamma_c \\ u = \Phi_i, \ \sigma(u)\cdot n = G_i & \text{on} \quad \Gamma_i \end{cases} \tag{3}$$

In order to solve this problem, we will use a (fictious) domain decompostion-like method [8, 9] which consists on splitting the problem (3) into two direct and well-posed problems using only one data on Γ_c.
Thus, let $(u_D^\lambda, p_D^\lambda)$ and $(u_N^\lambda, p_N^\lambda)$ be respectively the solution of the following Dirichlet and Neumann problems :

$$(\mathbb{P}_D) \begin{cases} -\nu\Delta u_D^\lambda + \nabla p_D^\lambda & = 0 & \text{in} & \Omega \\ \nabla \cdot u_D^\lambda & = 0 & \text{in} & \Omega \\ u_D^\lambda & = \Phi_c & \text{on} & \Gamma_c \\ u_D^\lambda & = \lambda & \text{on} & \Gamma_i \end{cases} \qquad (\mathbb{P}_N) \begin{cases} -\nu\Delta u_N^\lambda + \nabla p_N^\lambda & = 0 & \text{in} & \Omega \\ \nabla \cdot u_N^\lambda & = 0 & \text{in} & \Omega \\ (\sigma(u_N^\lambda)\cdot n)\cdot \tau & = g_c & \text{on} & \Gamma_c \\ u_N^\lambda \cdot n & = \Phi_c \cdot n & \text{on} & \Gamma_c \\ u_N^\lambda & = \lambda & \text{on} & \Gamma_i \end{cases}$$

A solution of the problem (2) is recovered if and only if the solutions of the well-posed above problems coïncide. The proposed data-recovering problem therefore amounts to minimizing the gap between u_D^λ and u_N^λ.
Following the study done in [10, 11], we define the cost function E which could be interpreted as an energy-type error functional. E is defined as follows :

$$E(\lambda) = \frac{1}{2}\int_\Omega \sigma(u_D^\lambda - u_N^\lambda) : \nabla(u_D^\lambda - u_N^\lambda) \tag{4}$$

We have proved in [7] the following proposition :

Proposition 1
1. E is a positive quadratic and convex functional on $(H^{\frac{1}{2}}(\Gamma_i))^2$.
2. For a compatible pair (Φ_c, g_c), the solution (Φ_i, G_i) of the partially overdetermined boundary value problem (2) is obtained by the following

$$\Phi_i = u_D^{\lambda_{min}}|_{\Gamma_i},\ G_i = (\sigma(u_N^{\lambda_{min}}) \cdot n)|_{\Gamma_i}$$

where λ_{min} is the solution of the following minimization problem :

$$\lambda_{min} = \arg \min_{\lambda \in (H^{\frac{1}{2}}(\Gamma_i))^2} E(\lambda) \tag{5}$$

2.1. Minimization procedure

We next prove the following result :

Proposition 2
For a compatible pair (Φ_c, g_c), the minimum of E is reached when :

$$\sigma(u_D^\lambda) \cdot n \quad = \sigma(u_N^\lambda) \cdot n \quad \text{on } \Gamma_i \tag{6}$$

Proof :
We derive the first optimality condition. It's easy to prove that for $h \in (H^{\frac{1}{2}}(\Gamma_i))^2$, we have :

$$\frac{\partial E}{\partial \lambda}(h) = \frac{1}{2} \int_\Omega \sigma(u_D^\lambda - u_N^\lambda) : \nabla(r_D^h - r_N^h)$$

where (r_D^h, s_D^h) and (r_N^h, s_N^h) are respectively the solutions of :

$$\left\{ \begin{array}{rlll} -\nu \Delta r_D^h + \nabla s_D^h & = 0 & \text{in} & \Omega \\ \nabla \cdot r_D^h & = 0 & \text{in} & \Omega \\ r_D^h & = 0 & \text{on} & \Gamma_c \\ r_D^h & = h & \text{on} & \Gamma_i \end{array} \right. , \qquad \left\{ \begin{array}{rlll} -\nu \Delta r_N^h + \nabla s_N^h & = 0 & \text{in} & \Omega \\ \nabla \cdot r_N^h & = 0 & \text{in} & \Omega \\ (\sigma(r_N^h) \cdot n) \cdot \tau & = 0 & \text{on} & \Gamma_c \\ r_N^h \cdot n & = 0 & \text{on} & \Gamma_c \\ r_N^h & = h & \text{on} & \Gamma_i \end{array} \right. \tag{7}$$

Green Formula gives :

$$\frac{\partial E}{\partial \lambda}(h) = \frac{1}{2} \int_{\partial\Omega} \left(\sigma(u_D^\lambda - u_N^\lambda) \cdot n\right) r_D^h - \frac{1}{2} \int_{\partial\Omega} \left(\sigma(r_N^h) \cdot n\right) (u_D^\lambda - u_N^\lambda)$$

since we have $r_D^h = 0$ on Γ_c and $u_D^\lambda - u_N^\lambda = 0$ on Γ_i, then :

$$\frac{\partial E}{\partial \lambda}(h) = \frac{1}{2} \int_{\Gamma_i} \left(\sigma(u_D^\lambda - u_N^\lambda) \cdot n\right) r_D^h - \frac{1}{2} \int_{\Gamma_c} \left(\sigma(r_N^h) \cdot n\right) (u_D^\lambda - u_N^\lambda)$$

using the boundary condition on $(u_D^\lambda - u_N^\lambda)\cdot n$ and on $(\sigma(r_N^h)\cdot n)\cdot \tau$, we conclude that :

$$\frac{\partial E}{\partial \lambda}(h) = \frac{1}{2}\int_{\Gamma_i} \left(\sigma(u_D^\lambda - u_N^\lambda)\cdot n\right) h, \ \ \forall h \in (H^{\frac{1}{2}}(\Gamma_i))^2.$$

thus our statement follows immediately.

2.2. The interfacial operators

Following the classical framework of the Domain Decomposition Community, we introduce the notations :

$$\begin{cases} (u_D^\lambda, p_D^\lambda) & = (u_D^0, p_D^0) + (r_D^\lambda, s_D^\lambda) \\ (u_N^\lambda, p_N^\lambda) & = (u_N^0, p_N^0) + (r_N^\lambda, s_N^\lambda) \end{cases}$$

thus, the condition (6) can be written as :

$$\sigma(r_D^\lambda)\cdot n - \sigma(r_N^\lambda)\cdot n = -[\sigma(u_D^0)\cdot n - \sigma(u_N^0)\cdot n]$$

or equivalently by using operator's modelling

$$S(\lambda) = T$$

with

$$T = -[\sigma(u_D^0)\cdot n - \sigma(u_N^0)\cdot n]$$

and $S = S_D - S_N$ is the Steklov-Poincaré operator defined by :

$$S(\lambda) = S_D(\lambda) - S_N(\lambda)$$

and where

$$\begin{array}{llll} S_D : & H^{1/2}(\Gamma_i)^2 & \to & H^{-1/2}(\Gamma_i)^2 \\ & \lambda & \to & \sigma(r_D^\lambda)\cdot n \end{array}, \qquad \begin{array}{llll} S_N : & H^{1/2}(\Gamma_i)^2 & \to & H^{-1/2}(\Gamma_i)^2 \\ & \lambda & \to & \sigma(r_N^\lambda)\cdot n \end{array} \tag{8}$$

2.3. Reconstruction of Robin coefficient

From the last equation in (1), we can now determine the value of the real parameter R using the means of the recovered values of u and $\sigma(u).n$ on Γ_i. More precisely, we use the formula :

$$|R| = \left| \frac{\int_{\Gamma_i}[\sigma(u_N).n]_1 + \int_{\Gamma_i}[\sigma(u_N).n]_2}{\int_{\Gamma_i}[u_N]_1 + \int_{\Gamma_i}[u_N]_2} \right| \tag{9}$$

where for a vector u of $\mathbb{R}^2$, $[u]_k$ denotes the k^{th} component of u.
We have not deal in the present work with the case of a spatially dependent R which will be treated later on.

3. Numerical Results

We use a numerical procedure based on the preconditioned gradient algorithm :

$$X_{k+1} = X_k - mP[S(X_k) - T]$$

where P is a preconditioning operator and m is a relaxation parameter. The expressions of S and T are described in the previous section.

3.1. Algorithm

1) Initialization : For $k = 0$ choose $\lambda_0 = 0$

2) Solve $(\mathbb{P}_D)$ and $(\mathbb{P}_N)$ whith $\lambda = \lambda_k$.

3) Compute w_k solution of the following "interface" problem :

$$(\mathbb{P}_I)\begin{cases} -\nu\Delta w_k + \nabla p_k &= 0 \quad \text{in} \quad \Omega \\ \nabla\cdot w_k &= 0 \quad \text{in} \quad \Omega \\ w_k &= 0 \quad \text{on} \quad \Gamma_c \\ \sigma(w_k)\cdot n &= \big(\sigma(u_D^k)\cdot n - \sigma(u_N^k)\cdot n\big) \quad \text{on} \quad \Gamma_i \end{cases} \tag{10}$$

4) Update λ :

$$\lambda_{k+1} = \lambda_k + m\, w_k$$

5) Stopping Criteria : $E(\lambda_k) < \varepsilon$, where ε is the tolerance (selected numerically).

6) Calculate R using formula (9)

3.2. Results and Discussions

We will test our method for two cases corresponding to different choices of the domain Ω. The first choice corresponds to an annular domain and the second to a rectangular one. The overdetermined data are generated from the following test examples given by [12, 9] and refered to by smooth and singular data respectively :

$$u(x,y) = (4y^3 - x^2, 4x^3 + 2xy - 1),\ \ p(x,y) = 24xy - 2x$$

$$\begin{cases} u(x,y) = \frac{1}{4\pi}\left(log\frac{1}{\sqrt{(x-a)^2+y^2}} + \frac{(x-a)^2}{(x-a)^2+y^2}, \frac{y(x-a)}{(x-a)^2+y^2}\right), \\ p(x,y) = \frac{1}{2\pi}\frac{x-a}{(x-a)^2+y^2}. \end{cases}$$

For each case and for different test values of R, we will compare the components of the velocity and those of the normal stress tensor for the analytical solution u_{exact}, u_D and u_N

on Γ_i. Then we will reconstruct on Γ_c the unkown values $(\sigma(u_D) \cdot n) \cdot n$, $(\sigma(u_N) \cdot n) \cdot n$ and compare them with $(\sigma(u_{exact}) \cdot n) \cdot n$.
Moreover, we will compare on Γ_i the normal stress of u_D and u_N with the limit condition $R\,u_{exact}$.
Finally, we will reconstruct the value of the Robin coefficient that we will call ρ and compare it with the exact used value R.
Computations are done under Freefem++ Software environment.

First example : Let Ω be the annular domain with radius $R_1 = 1$ and $R_2 = 2$. Γ_c will be the outer circle and Γ_i the inner one. we mesh with 150 nodes on Γ_c and 100 nodes on Γ_i. $\varepsilon = 6 \times 10^{-4}$ (80 iterations were required).
The reconstructed stress tensor on Γ_i from u_D and u_N are compared with the one from the exact solution (figure 1). We give the result for $R = 20$ but the numerical tests are done for several values of R and the results are satisfying.
In table 1 where we compare the exact value of the Robin coefficient R with the identified one by our method ρ, we note that the error rate is interesting it varies between 0.5% and 8.9%.

Second example : In this case, Ω is a rectangular domain with $L = 2$ and $\ell = 1$. $\partial\Omega = \Gamma_c \cup \Gamma_i \cup \Gamma_N$, where $\Gamma_c = [0,2] \times \{1\}$, $\Gamma_i = [0,2] \times \{0\}$, $\Gamma_N = (\{0\} \times [0,1]) \cup (\{2\} \times [0,1])$. We mesh with 60 nodes on Γ_c and Γ_i, and with 50 nodes on Γ_N. $\varepsilon = 3 \times 10^{-3}$ (50 iterations were required).

In figure 2 we plot the lacking component of the normal stress on Γ_c (left) and compare the normal stress with Ru_{exact} on Γ_i (right). Note that these reconstructed fields are in close agreement with the exact ones. We test for several values of R.

In table 2 we reconstruct the value of the Robin coefficient ρ and compare it with the exact one R. The error rate is varing between 1.2% and 7%.

In order to test the robustness of the used method, we introduce a white noise perturbation to the data with an amplitude ranging from 1 to 15%. We reconstruct the velocity and the stress tensor on Γ_i from these noisy data. We observe in figure3 that the method used is more robust with smooth data (left) than with singular one (right).

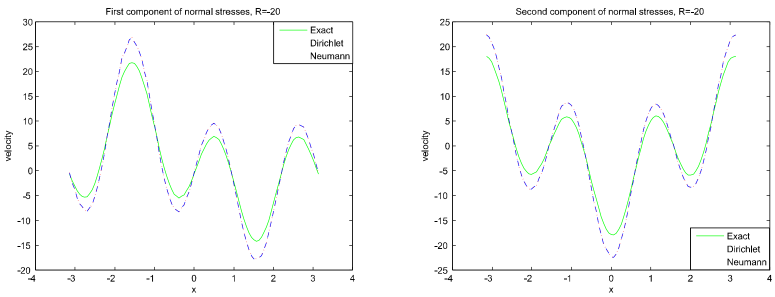

Figure 1. *First example with smooth data, R=20 : the reconstructed stress tensor on Γ_i*

Tableau 1. *First example : Comparaison of ρ and R*

R	5	10	50	70	100
ρ	5.07301	9.94297	45.5175	67.1686	93.8794

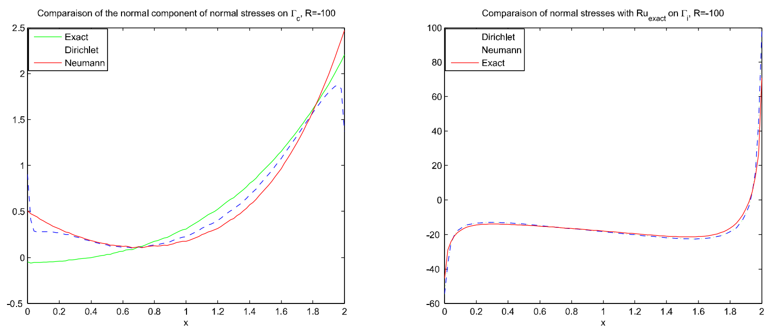

Figure 2. *Second example with smooth data, R=100 : the reconstructed data on Γ_c(left) and comparing normal stress with Ru_{exact} on Γ_i (right)*

Tableau 2. *Comparaison of ρ and R : Rectangular domain*

R	2	5	10	20	50	100
ρ	2.05149	4.93797	9.63617	18.8812	46.4296	92.9558

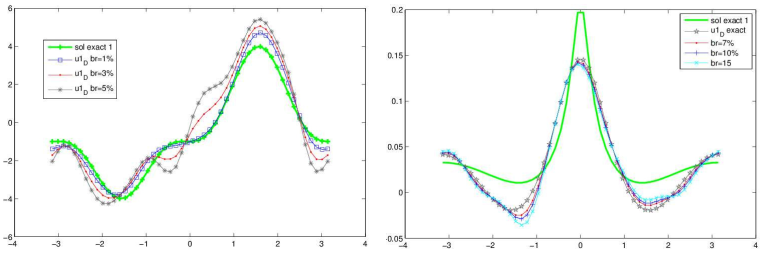

Figure 3. *Comparaison of velocity's first component for noisy data : Smooth data(left), Singular data with a=0.8 (right)*

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