Hopf bifurcation properties of a delayed Predator-Prey model with threshold prey harvesting

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RÉSUMÉ. Dans cet article, nous étudions les propriétés de la bifurcation de Hopf pour un modèle prédateur-proie à retard avec deux seuils de collecte des proies et la stabilité des solutions périodiques obtenues via la bifurcation de Hopf en utilisant la théorie des formes normales et la réduction sur la variété centrale pour les équations différentielles fonctionnelles retardées (EDFr). Le long de cet article, nous supposerons toujours que les équations subissent une bifurcation de Hopf à l'équilibre positif $G(x^*, y^*)$ pour $\tau = \tau_0^j$,$(j = 0, 1, 2, ...)$ et les $\pm i\omega_0$ correspondent aux racines imaginaires pures de l'équation caractéristique.

ABSTRACT. In this paper,we shall study the properties of the Hopf bifurcations obtained for a delayed predator-prey model with threshold prey harvesting and the stability of bifurcated periodic solutions occurring through Hopf bifurcation by using the normal form theory and the center manifold reduction for retarded functional dfferential equations(RFDEs). Throughout this paper,we always assume that the equations undergoes Hopf bifurcation at the positive equilibrium $G(x^*, y^*)$ for $\tau = \tau_0^j$,$(j = 0, 1, 2, ...)$ and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation.

MOTS-CLÉS : Retard; prédateur-proie; bifurcation de hopf; bifurcations locales.

KEYWORDS : Delay; predator-prey; Hopf bifurcation; local bifurcations.

1. Introduction

In this paper, we consider a system of delayed differential equations modelling the predator-prey dynamic with a continuous double thresholds harvesting and a Holling response function of type III. Recently, Tankam & al. [3] considered the following model :

$$\begin{cases} \dot{x}(t) &= \varphi(x(t)) - my(t)p(x(t)) - H(x(t)), \\ \dot{y}(t) &= [-d + cmp(x(t-\tau))]y(t). \end{cases} \quad [1]$$

where x and y represent the population of preys and predators respectively. d is the natural mortality rate of the predators. c and m are positive constants. The function

$$\varphi(x) = rx\left(1 - \frac{x}{K}\right), \quad [2]$$

models the dynamics of preys in absence of predators, r is the growth rate of preys for small values of x, while K is the capacity of the environment to support the preys. The function $p(x)$ is the Holling response function of type III given by :

$$p(x) = \frac{x^2}{a\,x^2 + b\,x + 1}, \quad [3]$$

(where $a > 0$ is constant and b is nonnegative constant)
and $H(x)$ is the double thresholds harvesting function given by :

$$H(x) = \begin{cases} 0 & \text{if } x < T_1, \\ \dfrac{h(x-T_1)}{T_2 - T_1} & \text{if } T_1 \leq x \leq T_2, \\ h & \text{if } x \geq T_2, \end{cases} \quad [4]$$

This piecewise linear operator policy harvesting has been introduced in [1] in a predator-prey model without delay, where a Holling response function of type II was considered. In 2015, Tankam & al. have prooved that a Hopf bifurcation occurs. The following Theorem was given :

Theorem 1 *(Tankam & al., 2015) Suppose that a positive equilibrium E exists and is locally asymptotically stable for (1) with $\tau = 0$. Also let $\eta_0 = w_0^2$ be a positive root of $\eta^2 + \left[\varphi'(x^\star) - H'(x^\star) - mp'(x^\star)y^\star\right]^2 \eta - dmp'(x^\star)y^{\star 2} = 0$. Then there exists a $\tau = \tau^0$ such that E is locally asymptotically stable for $\tau \in (0, \tau^0]$ and unstable for $\tau > \tau^0$. Furthermore, the system undergoes a Hopf bifurcation at E when $\tau = \tau^0$.*

The aim of the following section is to study the properties of the Hopf bifurcation obtained by Theorem 1 and stability of bifurcated periodic solutions occuring through the Hopf bifurcation.

2. Properties of Hopf Bifurcation

In this section, we analyse the properties of the Hopf bifurcation using normal forms theory as in Hassard et al.[2]. The main result is given in Theorem 2 after having been

proved by pre-calculations.
Considering the equations (1) and $x_1(t) = x(t) - x^\star$ and $x_2(t) = y(t) - y^\star$; then system (1) is equivalent to the following two dimensional system :

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & \Big[\varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star)\Big]\, x_1(t) - mp(x^\star)\, x_2(t) + f_1\big(x_1(t), x_2(t)\big), \\ \dot{x}_2(t) & = & cmy^\star p'(x^\star)\, x_1(t-\tau) + f_2\big(x_1(t), x_2(t), x_1(t-\tau)\big). \end{array}\right. \quad [5]$$

where

$$\begin{array}{rcl} f_1\big(x_1(t), x_2(t)\big) & = & \varphi\big(x_1(t) + x^\star\big) - m(x_2(t) + y^\star)p\big(x_1(t) + x^\star\big) - H\big(x_1 + x^\star\big) \\ & - & \Big[\varphi'(x^\star) - my^\star(t)p'(x^\star) - H'(x^\star)\Big]\, x_1(t) + mp(x^\star)\, x_2(t) \end{array}$$

and

$$\begin{array}{rcl} f_2\big(x_1(t), x_2(t), x_1(t-\tau)\big) & = & \Big[- d + cmp\big(x_1(t-\tau) + x^\star\big)\Big]\big(x_2(t) + y^\star\big) \\ & - & y^\star cmp\big(x^\star\big)x_1(t-\tau) \end{array}$$

let $\tau = \tau_j^0 + \mu$; then $\mu = 0$ is the Hopf bifurcation value of system (1) at the positive equilibrium $G(x^\star, y^\star)$. Since system (1) is equivalent to system (5), in the following discussion we shall consider mainly system (5).

In system (5), let $\bar{x}_k(t) = x_k(\tau t)$ and drop the bars for simplicity of notation. Then system (5) can be rewritten as a system of RFDEs in $\mathcal{C}\big([-1, 0], R^2\big)$ of the form :

$$\left\{\begin{array}{rcl} \dot{x}_1(t) & = & (\tau_j^0 + \mu)\Big[\varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star)\Big]\, x_1(t) - (\tau_j^0 + \mu)mp(x^\star)\, x_2(t) \\ & + & (\tau_j^0 + \mu)f_1\big(x_1(t), x_2(t)\big), \\ \dot{x}_2(t) & = & (\tau_j^0 + \mu)cmy^\star p'(x^\star)\, x_1(t-\tau) + (\tau_j^0 + \mu)f_2\big(x_1(t), x_2(t), x_1(t-\tau)\big). \end{array}\right. \quad [6]$$

Let us consider the following lemma proved in annex.

Lemma 1 *The system [6] is equivalent to*

$$\dot{x}(t) = A(\mu)x_t + R(\mu)x_t, \quad [7]$$

where $A(\mu)$ is linear. Besides, there exists an inner product $< \bullet, \bullet >$ and eigenvectors $q(\theta)$ and $q^\star(s)$ respectively of $A(0)$ and $A^\star$ such as $< q^\star(s), q(\theta) >= 1$, where $A^\star$ is the associate operator of A.

Using the same notations as in [2], we first compute the coordinates to describe the center manifold $\mathcal{C}_0$ at $\mu = 0$. Let x_t be the solution of Equation (5) when $\mu = 0$. Define

$$\begin{array}{rcl} z(t) & = & \langle q^* \ , \ x_t \rangle \\ W(t, \theta) & = & x_t(\theta) - 2\mathcal{R}_e\Big(z(t)q(\theta)\Big) \\ & = & x_t(\theta) - \Big(z(t)q(\theta) + \bar{z}(t)\bar{q}(\theta)\Big) \end{array} \quad [8]$$

On the center manifold $\mathcal{C}_0$ we have

$$W(t, \theta) = W(z, \bar{z}, \theta) \quad [9]$$

where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + W_{30}(\theta)\frac{z^3}{6} + \cdots \qquad [10]$$

z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Note that W is real if x_t is real. We only consider real solutions. For solution $x_t \in \mathcal{C}_0$ of (5), since $\mu = 0$, we have

$$\dot{z}(t) = iw_0\tau_j^0 z + \bar{q}^*(0)f\Big(0, W(z,\bar{z},0) + 2\mathcal{R}_e\big(z(t)q(\theta)\big)\Big) \equiv iw_0\tau_j^0 z + \bar{q}^*(0)f_0(z,\bar{z})$$

We rewrite this equation as

$$\dot{z}(t) = iw_0\tau_j^0 z + g\big(z,\bar{z}\big) \qquad [11]$$

where

$$g\big(z,\bar{z}\big) = g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots \qquad [12]$$

The following lemma gives the values of the coefficients of $g(z,\bar{z})$.

Lemma 2

$$\begin{aligned}
g_{20} &= 2\tau_j^0\bar{D}\Big[-\Big(\tfrac{r}{K} + mp'(x^\star)\nu_1 + \tfrac{my^\star p''(x^\star)}{2}\Big)\\
&+ \bar{\nu}_1\left(\tfrac{y^\star cmp''(x^\star)e^{-2iw_0\tau_j^0}}{2} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0}\right)\\
g_{02} &= 2\tau_j^0\bar{D}\Big[-\Big(\tfrac{r}{K} + mp'(x^\star)\bar{\nu}_1 + \tfrac{my^\star p''(x^\star)}{2}\Big)\\
&+ \bar{\nu}_1\left(\tfrac{y^\star cmp''(x^\star)e^{2iw_0\tau_j^0}}{2} + cmp'(x^\star)\bar{\nu}_1 e^{iw_0\tau_j^0}\right)\\
g_{11} &= 2\tau_j^0\bar{D}\Big[-\Big(\tfrac{r}{K} + mp'(x^\star)\mathcal{R}_e\{\nu_1\} + \tfrac{my^\star p''(x^\star)}{2}\Big)\\
&+ \bar{\nu}_1\Big(\tfrac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\Big\{\nu_1 e^{iw_0\tau_j^0}\Big\}\Big)\\
g_{21} &= \tau_j^0\bar{D}\Big[-\tfrac{r}{K}\Big(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\Big)\\
&- mp'(x^\star)\Big(2W_{11}^{(2)}(0) + W_{20}^{(2)}(0) + \bar{\nu}_1 W_{20}^{(1)}(0) + 2\nu_1 W_{11}^{(1)}(0)\Big)\\
&- \tfrac{mp''(x^\star)}{2}(2\bar{\nu}_1 + 4\nu_1) - \tfrac{my^\star p''(x^\star)}{2}\Big(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\Big)\\
&+ \bar{\nu}_1 my^\star p''(x^\star)\Big(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1)e^{iw_0\tau_j^0}\Big)\\
&+ \bar{\nu}_1 cmp'(x^\star)\Big(\bar{\nu}_1 W_{20}^{(1)}(-1) + W_{20}^{(2)}(0)e^{iw_0\tau_j^0} + 2W_{11}^{(2)}(0)e^{-iw_0\tau_j^0} + 2\nu_1 W_{11}^{(1)}(-1)\Big)\\
&+ \tfrac{cmp''(x^\star)}{2}\Big(4\nu_1 + 2\bar{\nu}_1 e^{-2iw_0\tau_j^0}\Big)\Big]
\end{aligned}$$

[13]

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From (35) (*cf Annex 1*) and (8), we have :

$$\begin{aligned}\dot{W} &= \dot{x}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2\mathcal{R}_e\Big\{\bar{q}^*(0)f_0q(\theta)\Big\}, & \theta \in [-1;0)\,; \\ AW - 2\mathcal{R}_e\Big\{\bar{q}^*(0)f_0q(\theta)\Big\} + f_0, & \theta = 0. \end{cases} \\ &\equiv^{\text{def}} AW + \mathcal{H}\big(z,\bar{z},\theta\big)\end{aligned} \quad [14]$$

where

$$\mathcal{H}\big(z,\bar{z},\theta\big) = \mathcal{H}_{20}(\theta)\frac{z^2}{2} + \mathcal{H}_{11}(\theta)z\bar{z} + \mathcal{H}_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \quad [15]$$

Substituting the corresponding series into (14) and comparing the coefficients, we obtain

$$\begin{aligned}(A - 2iw_0\tau_j^0)W_{20}(\theta) &= -\mathcal{H}_{20}(\theta) \\ AW_{11}(\theta) &= -\mathcal{H}_{11}(\theta)\end{aligned} \quad [16]$$

From (14), we know that for $\theta \in [-1,0)$,

$$\mathcal{H}\big(z,\bar{z},\theta\big) = -\bar{q}^*(0)f_0q(\theta) - q^*(0)\bar{f}_0\bar{q}(\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \quad [17]$$

Comparing the coefficient with (15), we get :

$$-g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) = H_{20}(\theta) \quad [18]$$

$$-g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) = H_{11}(\theta) \quad [19]$$

From (16) and (18) and the definition of A, it follows that

$$\dot{W}(\theta) = 2iw_0\tau_j^0W_{20} + g_{20}q(\theta) + \bar{g}_{02}\bar{q}(\theta) \quad [20]$$

Notice that $q(\theta) = \Big(1,\nu_1\Big)^T e^{iw_0\tau_j^0\theta}$. Hence,

$$W_{20}(\theta) = \frac{ig_{20}}{w_0\tau_j^0}q(0)e^{iw_0\tau_j^0\theta} + \frac{i\bar{g}_{02}}{3w_0\tau_j^0}\bar{q}(0)e^{-iw_0\tau_j^0\theta} + E_1e^{2iw_0\tau_j^0\theta} \quad [21]$$

where $E_1 = \Big(E_1^{(1)}, E_1^{(2)}\Big) \in R^2$ is a constant vector. Similarly, from (16) and (19), we obtain

$$W_{11}(\theta) = -\frac{ig_{11}}{w_0\tau_j^0}q(0)e^{iw_0\tau_j^0\theta} + \frac{i\bar{g}_{11}}{w_0\tau_j^0}\bar{q}(0)e^{-iw_0\tau_j^0\theta} + E_2 \quad [22]$$

where $E_2 = \Big(E_2^{(1)}, E_2^{(2)}\Big) \in R^2$ is also a constant vector.

In what follows, we will seek appropriate E_1 and E_2. From the definition of A and (16), we obtain

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2iw_0\tau_jW_{20}(0) - H_{20}(0) \quad [23]$$

$$\int_{-1}^{0} d\eta(\theta) W_{11}(\theta) = -H_{11}(0) \qquad [24]$$

where $\eta(\theta) = \eta(0,\theta)$. By (14), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_j^0 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{pmatrix} \qquad [25]$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_j^0 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} \end{pmatrix} \qquad [26]$$

Substituting (21) and (25) into (23) and noticing that

$$\begin{array}{rcl} \left(iw_0\tau_j^0 I - \int_{-1}^0 e^{iw_0\tau_j^0\theta} d\eta(\theta)\right) q(0) & = & 0 \\ \left(-iw_0\tau_j^0 I - \int_{-1}^0 e^{-iw_0\tau_j^0\theta} d\eta(\theta)\right) \bar{q}(0) & = & 0 \end{array} \qquad [27]$$

we obtain

$$\left(2iw_0\tau_j^0 I - \int_{-1}^0 e^{2iw_0\tau_j^0\theta} d\eta(\theta)\right) E_1 =$$

$$2\tau_j^0 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{pmatrix}$$

This leads to

$$\begin{pmatrix} 2iw_0 - \varphi'(x^\star) + my^\star p'(x^\star) + H'(x^\star) & mp(x^\star) \\ y^\star cmp'(x^\star) e^{-2iw_0\tau_j^0} & 2iw_0 \end{pmatrix} E_1$$

$$= 2 \begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{pmatrix}$$

Solving this system for E_1, we obtain

$$E_1^{(1)} = \frac{2}{\sigma} \begin{vmatrix} -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} & mp(x^\star) \\ \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} & 2iw_0 \end{vmatrix}$$

$$E_1^{(2)} =$$

$$\frac{2}{\sigma} \begin{vmatrix} 2iw_0 - \varphi'(x^\star) + my^\star p'(x^\star) + H'(x^\star) & -\frac{r}{K} - mp'(x^\star)\nu_1 - \frac{my^\star p''(x^\star)}{2} \\ y^\star cmp'(x^\star) e^{-2iw_0\tau_j^0} & \frac{y^\star cmp''(x^\star)}{2} e^{-2iw_0\tau_j^0} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0} \end{vmatrix}$$

where

$$\sigma = \begin{vmatrix} 2iw_0 - \varphi'(x^\star) + my^\star p'(x^\star) + H'(x^\star) & mp(x^\star) \\ y^\star cmp'(x^\star)e^{-2iw_0\tau_j^0} & 2iw_0 \end{vmatrix}$$

Similarly, substituting (22) and (26) into (24), we get

$$\begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ -y^\star cmp'(x^\star) & 0 \end{pmatrix} E_2$$

$$= 2\begin{pmatrix} -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} \\ \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} \end{pmatrix}$$

and hence

$$E_2^{(1)} = \frac{2}{\varrho}\begin{vmatrix} -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} & -mp(x^\star) \\ \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} & 0 \end{vmatrix}$$

$$E_2^{(2)} = \frac{2}{\varrho}\begin{vmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -\frac{r}{K} - mp'(x^\star)\mathcal{R}_e\{\nu_1\} - \frac{my^\star p''(x^\star)}{2} \\ -y^\star cmp'(x^\star) & \frac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\left\{\nu_1 e^{iw_0\tau_j^0}\right\} \end{vmatrix}$$

where

$$\varrho = \begin{vmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ -y^\star cmp'(x^\star) & 0 \end{vmatrix}$$

Thus, we can determine W_{20} and W_{11} from (21) and (22). Furthermore, g_{21} in (13) can be expressed by the parameters and delay. Thus, we can compute the following values :

$$\begin{aligned} C_1(0) &= \frac{i}{2w_0\tau_j^0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2} \\ \nu_2 &= -\frac{\mathcal{R}_e\{C_1(0)\}}{\mathcal{R}_e\left\{\lambda'(\tau_j^0)\right\}} \\ \beta_2 &= 2\mathcal{R}_e\{C_1(0)\} \\ P_2 &= -\frac{\mathcal{I}_m\{C_1(0)\} + \nu_2\mathcal{I}_m\left\{\lambda'(\tau_j^0)\right\}}{w_0\tau_j^0} \end{aligned} \qquad [28]$$

which determine the qualities of bifurcating periodic solution in the center manifold at the critical value τ_j^0.

Theorem 2 *: In Eq. (28), the sign of ν_2 determines the direction of the Hopf bifurcation. Thus, if $\nu_2 > 0$, then the Hopf bifurcation is supercritical and the bifurcating periodic* *solution exists for $\tau_1 > \tau_1^0$. If $\nu_2 < 0$, then the Hopf bifurcation is subcritical and the bifurcating periodic solution exists for $\tau_1 < \tau_1^0$. β_2 determines the stability of the bifurcating periodic solution : The bifurcating periodic solutions are stable if $\beta_2 < 0$ and unstable if $\beta_2 > 0$. T_2 determines the period of the bifurcating periodic solutions : the period increases if $P_2 > 0$ and decreases if $P_2 < 0$.*

3. Bibliographie

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Annex 1 : Proof of the Lemma 1

Define the linear operator $L(\mu) : \mathcal{C} \to R^2$ and the nonlinear operator $f(\cdot, \mu) : \mathcal{C} \to R^2$ by :

$$
\begin{array}{rcl}
L_\mu(\phi) & = & (\tau_j^0 + \mu) \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} \\
& + & (\tau_j^0 + \mu) \begin{pmatrix} 0 & 0 \\ y^\star cmp'(x^\star) & 0 \end{pmatrix} \begin{pmatrix} \phi_1(-1) \\ \phi_2(-1) \end{pmatrix}
\end{array} \quad [29]
$$

and

$$
f(\phi, \mu) = (\tau_j^0 + \mu) \begin{pmatrix} f_1\big(\phi_1(0), \phi_2(0)\big) \\ f_2\big(\phi_1(0), \phi_2(0), \phi_1(-1)\big) \end{pmatrix} \quad [30]
$$

respectively, where $\phi = (\phi_1, \phi_2)^T \in \mathcal{C}$.

By the Riesz representation theorem, there exists a 2×2 matrix function $\eta(\theta, \mu)$, $-1 \leq \theta \leq 0$ whose elements are of bounded variation such that

$$
L_\mu(\phi) = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta) \quad \text{for} \quad \phi \in \mathcal{C}\big([-1, 0], R^2\big). \quad [31]
$$

In fact, we can choose

$$
\begin{array}{rcl}
\eta(\theta, \mu) & = & (\tau_j^0 + \mu) \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} \delta\big(\theta\big) \\
& + & (\tau_j^0 + \mu) \begin{pmatrix} 0 & 0 \\ y^\star cmp'(x^\star) & 0 \end{pmatrix} \delta\big(\theta + 1\big)
\end{array} \quad [32]
$$

where δ is the Dirac delta function

For $\phi \in \mathcal{C}\big([-1, 0], R^2\big)$, define

$$
A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0)\,; \\ \int_1^0 d\eta(\mu, s)\phi(s), & \theta = 0. \end{cases} \quad [33]
$$

and

$$
R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0)\,; \\ f(\mu, \phi), & \theta = 0. \end{cases} \quad [34]
$$

Then, the system (6) is equivalent to

$$
\dot{x}(t) = A(\mu)x_t + R(\mu)x_t \quad [35]
$$

where $x_t(\theta) = x(t + \theta)$, $\theta \in [-1, 0]$.

For $\psi \in \mathcal{C}^1\big([0, 1], R^2\big)$, define

$$A^*\psi = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0,1]\,; \\ \int_1^0 d\eta(t,0)\phi(-t), & s = 0. \end{cases} \qquad [36]$$

and a bilinear inner product

$$\langle \psi(s), \phi(\theta)\rangle = \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi \qquad [37]$$

where $\eta(\theta) = \eta(\theta, 0)$. In addition, by Theorem 1 we know that $\pm i w_0 \tau_j^0$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^*. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to $iw_0\tau_j^0$ and $q^*(s)$ be the eigenvector of A^* corresponding to $-iw_0\tau_j^0$.

Let $q(\theta) = \begin{pmatrix} 1, & \nu_1 \end{pmatrix}^T e^{iw_0\tau_j^0\theta}$ and $q^*(s) = D\begin{pmatrix} 1, & \nu_1^* \end{pmatrix}^T e^{iw_0\tau_j^0 s}$. From the above discussion, it is easy to know that $A(0)q(0) = iw_0\tau_j^0 q(0)$ and $A^*(0)q^*(0) = -iw_0\tau_j^0 q^*(0)$. That is

$$\tau_j^0 \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} q(0)$$
$$+\tau_j^0 \begin{pmatrix} 0 & 0 \\ y^\star cmp'(x^\star) & 0 \end{pmatrix} q(-1) = iw_0\tau_j^0 q(0)$$

and

$$\tau_j^0 \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & 0 \\ -mp(x^\star) & 0 \end{pmatrix} q^*(0)$$
$$+\tau_j^0 \begin{pmatrix} 0 & y^\star cmp'(x^\star) \\ 0 & 0 \end{pmatrix} q^*(-1) = -iw_0\tau_j^0 q^*(0)$$

Thus, we can easily obtain

$$q(\theta) = \left(1\,, \quad \frac{y^\star cmp'(x^\star)e^{-iw_0\tau_j^0}}{iw_0}\right)^T e^{iw_0\tau_j^0\theta} \qquad [38]$$

$$q^*(s) = D\left(1\,, \quad \frac{mp(x^\star)}{iw_0}\right)^T e^{iw_0\tau_j^0 s} \qquad [39]$$

In order to assure $\langle \bar{q}^*(s), q(\theta)\rangle = 1$, we need to determine the value of D. From (37), we have

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle &= \bar{q}^*(0)q(0) - \textstyle\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi-\theta)d\eta(\theta)q(\xi)d\xi \\ &= \bar{q}^*(0)q(0) - \textstyle\int_{-1}^{0}\int_{\xi=0}^{\theta} \bar{D}\Big(1,\ \bar{\nu}_1^*\Big)e^{-iw_0\tau_j^0(\xi-\theta)}d\eta(\theta)\Big(1,\ \nu_1\Big)^T e^{iw_0\tau_j^0\xi}d\xi \\ &= \bar{q}^*(0)q(0) - \bar{q}^*(0)\textstyle\int_{-1}^{0}\theta e^{iw_0\tau_j^0\theta}d\eta(\theta)q(0) \\ &= \bar{q}^*(0)q(0) \\ &\quad -\bar{q}^*(0)\tau_j^0 \begin{pmatrix} \varphi'(x^\star) - my^\star p'(x^\star) - H'(x^\star) & -mp(x^\star) \\ 0 & 0 \end{pmatrix} \\ &\quad \Big(-e^{-iw_0\tau_j^0}\Big)q(0) \\ &= \bar{D}\Big[1 + \nu_1\bar{\nu}_1^* + \tau_j^0 e^{-iw_0\tau_j^0\bar{\nu}_1^*}y^\star cmp'(x^\star)\Big] \end{aligned}$$

So, we have

$$\begin{aligned} \bar{D} &= \frac{1}{1+\nu_1\bar{\nu}_1^*+\tau_j^0 e^{-iw_0\tau_j^0\bar{\nu}_1^*} y^\star cmp'(x^\star)} \\ D &= \frac{1}{1+\bar{\nu}_1\nu_1^*+\tau_j^0 e^{iw_0\tau_j^0\nu_1^*} y^\star cmp'(x^\star)} \end{aligned} \qquad [40]$$

That ends our proof.

Annex 2 : Proof of the Lemma 2

We have $x_t(\theta) = \big(x_{1t}(\theta), x_{2t}(\theta)\big)$ and $q(\theta) = \big(1, \nu_1\big)^T e^{iw_0\tau_j^0\theta}$. So, from (8) and (10), it follows that

$$\begin{aligned} x_t(\theta) &= W(t,\theta) + 2\mathcal{R}_e\Big(z(t)q(\theta)\Big) \\ &= W_{20}(\theta)\tfrac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}\tfrac{\bar{z}^2}{2} + \big(1,\nu_1\big)^T e^{iw_0\tau_j^0\theta} z(t) + \big(1,\bar{\nu}_1\big)^T e^{-iw_0\tau_j^0\theta}\bar{z}(t) + \cdots \end{aligned} \qquad [41]$$

and then we have

$$\begin{aligned} x_{1t}(0) &= z + \bar{z} + W_{20}^{(1)}(0)\tfrac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\tfrac{\bar{z}^2}{2} + \cdots \\ x_{2t}(0) &= \nu_1 z + \bar{\nu}_1\bar{z} + W_{20}^{(2)}(0)\tfrac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\tfrac{\bar{z}^2}{2} + \cdots \\ x_{1t}(-1) &= ze^{-iw_0\tau_j^0} + \bar{z}e^{iw_0\tau_j^0} + W_{20}^{(1)}(-1)\tfrac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\tfrac{\bar{z}^2}{2} + \cdots \\ x_{2t}(-1) &= \nu_1 ze^{-iw_0\tau_j^0} + \bar{\nu}_1\bar{z}e^{iw_0\tau_j^0} + W_{20}^{(2)}(-1)\tfrac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z} + W_{02}^{(2)}(-1)\tfrac{\bar{z}^2}{2} + \cdots \end{aligned} \qquad [42]$$

It follows together with (30) that

$$\begin{aligned} g(z,\bar{z}) &= \tfrac{z^2}{2}\Big\{2\tau_j^0\bar{D}\Big[-\Big(\tfrac{r}{K} + mp'(x^\star)\nu_1 + \tfrac{my^\star p''(x^\star)}{2}\Big) \\ &+ \bar{\nu}_1\Big(\tfrac{y^\star cmp''(x^\star)e^{-2iw_0\tau_j^0}}{2} + cmp'(x^\star)\nu_1 e^{-iw_0\tau_j^0}\Big)\Big]\Big\} \\ &+ \tfrac{\bar{z}^2}{2}\Big\{2\tau_j^0\bar{D}\Big[-\Big(\tfrac{r}{K} + mp'(x^\star)\bar{\nu}_1 + \tfrac{my^\star p''(x^\star)}{2}\Big) \\ &+ \bar{\nu}_1\Big(\tfrac{y^\star cmp''(x^\star)e^{2iw_0\tau_j^0}}{2} + cmp'(x^\star)\bar{\nu}_1 e^{iw_0\tau_j^0}\Big)\Big]\Big\} \\ &+ z\bar{z}\Big\{2\tau_j^0\bar{D}\Big[-\Big(\tfrac{r}{K} + mp'(x^\star)\mathcal{R}_e\{\nu_1\} + \tfrac{my^\star p''(x^\star)}{2}\Big) \\ &+ \bar{\nu}_1\Big(\tfrac{y^\star cmp''(x^\star)}{2} + cmp'(x^\star)\mathcal{R}_e\Big\{\nu_1 e^{iw_0\tau_j^0}\Big\}\Big)\Big]\Big\} \\ &+ \tfrac{z^2\bar{z}}{2}\Big\{\tau_j^0\bar{D}\Big[-\tfrac{r}{K}\Big(4W_{11}^{(1)}(0) + 2W_{20}^{(1)}(0)\Big) \end{aligned}$$

$$
\begin{aligned}
- &\quad mp'(x^\star)\Big(2W_{11}^{(2)}(0)+W_{20}^{(2)}(0)+\bar{\nu}_1 W_{20}^{(1)}(0)+2\nu_1 W_{11}^{(1)}(0)\Big)\\
- &\quad \tfrac{mp''(x^\star)}{2}(2\bar{\nu}_1+4\nu_1)-\tfrac{my^\star p''(x^\star)}{2}\Big(4W_{11}^{(1)}(0)+2W_{20}^{(1)}(0)\Big)\\
+ &\quad \bar{\nu}_1 my^\star p''(x^\star)\Big(2W_{11}^{(1)}(-1)+W_{20}^{(1)}(-1)e^{iw_0\tau_j^0}\Big)\\
+ &\quad \bar{\nu}_1 cmp'(x^\star)\Big(\bar{\nu}_1 W_{20}^{(1)}(-1)+W_{20}^{(2)}(0)e^{iw_0\tau_j^0}\\
&\quad +2W_{11}^{(2)}(0)e^{-iw_0\tau_j^0}+2\nu_1 W_{11}^{(1)}(-1)\Big)\\
+ &\quad \tfrac{cmp''(x^\star)}{2}\Big(4\nu_1+2\bar{\nu}_1 e^{-2iw_0\tau_j^0}\Big)\Big]\Big\}
\end{aligned}
$$

Where f and D are given in the proof of the lemma 1 respectively by (30) and (40).

Comparing the coefficients with (12), we obtain the coefficients of $g(z,\bar{z})$.
That ends our proof.