Rubrique

Identification of source for the bidomain equation using topological gradient

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RÉSUMÉ. Nous présentons une approche pour estimer les sources électriques dans le coeur à partir de mesures non invasives enregistrées sur la surface externe du thorax. L'approche est basée sur la méthode du gradient topologique. Cette méthode consiste à étudier le comportement d'une fonction coût via une perturbation locale du domaine. Nous montrons que l'approche proposée est capable d'identifier un terme source quand le support de la source est réduit dans l'espace.

ABSTRACT. We present an approach for estimating electrical sources within the heart domain from noninvasive measurements recorded on the outer surface of the torso. The approach is based on the topological gradient method. This method studies the behavior of a cost function during a local perturbation of the domain. We show that the proposed approach based on the topological gradient method has actually been able to identify the source terms when they are clustred in space.

MOTS-CLÉS : Le modèle bidomaine, électrophysiologie cardiaque, gradient topologique, analyse de sensibilité.

KEYWORDS : Bidomain model, cardiac electrophsiology , topological gradient, analysis sensibility.

1 Introduction

In order to localize the electrical sources in the heart, we make use of a recent method based on the topological gradient introduced by Sokolowski [7] and Masmoudi [6]. The topological gradient was originally used as part of the optimization shapes in solid mechanics [5]. This approach has subsequently been applied to a large number of areas : in imaging, it was first used for the detection of contours [4], in image classification [1], inpainting [2] and segmentation [3]. The calculation of topological sensitivity associated with the cost function of the inverse problem provides good qualitative information on the location of obstacles.

In this work, we are interested in the identification of the source term f from the boundary data obtained from the solution of the following system of equations :

$$\begin{cases} -div((\boldsymbol{\sigma}_\mathbf{i}+\boldsymbol{\sigma}_\mathbf{e})\nabla u_e) & = f & \text{in } \Omega_H \\ -div(\boldsymbol{\sigma}_\mathbf{T}\nabla u_T) & = 0 & \text{in } \Omega_T \\ \boldsymbol{\sigma}_\mathbf{T}\nabla u_T.n_T & = 0 & \text{on } \Gamma_{\text{ext}}. \\ u_e & = u_T & \text{on } \Sigma, \\ \boldsymbol{\sigma}_\mathbf{e}\nabla u_e.n+\boldsymbol{\sigma}_\mathbf{T}\nabla u_T.n_T & = 0 & \text{on } \Sigma, \end{cases} \quad (1)$$

where Ω_H (respectively Ω_T) is the heart (respectively, torso) domain (see figure 1), $\Sigma = \partial\Omega_H$ is the epicardial boundary and Γ_{ext} is the body surface. The tensors $\boldsymbol{\sigma}_\mathbf{i}, \boldsymbol{\sigma}_\mathbf{e}$ and $\boldsymbol{\sigma}_\mathbf{T}$ are respectively the intracellular, extracellular and thoracic conductivity tensors. The torso potential is denoted by u_T. The source term f is defined by

$$f = div(\boldsymbol{\sigma}_\mathbf{i}\nabla V_m)$$

where $V_m = u_i - u_e$ with u_e and u_i are respectively the extra-cellular potential and the intra-cellular potential. If we consider the dynamic of the electrical wave, the transmembrane potential V_m is governed by a reaction diffusion equation and is coupled to the extra-cellular potential, following these equations

$$\begin{cases} \chi_m\partial_t V_m + I_{ion}(V_m,w) - div(\boldsymbol{\sigma}_\mathrm{i}\nabla V_m) - div(\boldsymbol{\sigma}_\mathrm{i}\nabla u_e) & = I_{app} & \text{in } \Omega\times(0,T), \\ \partial_t w + G(V_m,w) & = 0 & \text{in } \Omega\times(0,T) \\ \boldsymbol{\sigma}_\mathrm{i}\nabla V_m.n & = 0 & \text{on } \partial\Omega\times(0,T), \end{cases} \quad (2)$$

where I_{app} is a given external current stimulus. w represents the concentrations of different chemical species and variables representing the openings or closures of some gates of the ionic channels. The ionic current $I_{ion}(V_m,w)$ and the function $G(V_m,w)$ are described by the Mitchell and Schaeffer model [8]. Note that the equation (1) represents the diffusion of the electrical potential at a given time. The combination of equations (1) and (2) provides the model of the electrical wave propagation in the heart and the torso. This is known in the litterature as the the bidomain-torso coupled problem. In this study, the dynamic of the electrical wave is not considered in the identification of the source, we only consider (1). The bidomain-torso coupled problem is only used to generate synthetical observations.

By defining $\Omega = \Omega_H \cup \Omega_T$, $u = \begin{cases} u_e & \text{in } \Omega_H \\ u_T & \text{in } \Omega_T \end{cases}$ and $\boldsymbol{\sigma} = \begin{cases} (\boldsymbol{\sigma}_\mathrm{i}+\boldsymbol{\sigma}_\mathbf{e}) & \text{in } \Omega_H \\ \boldsymbol{\sigma}_\mathbf{T} & \text{in } \Omega_T \end{cases}$, the problem (1) could be rewritten as follows

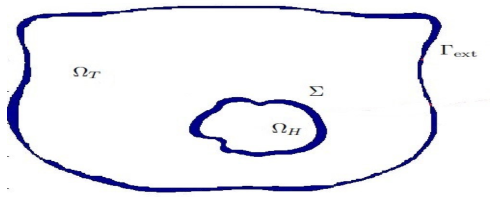

Figure 1 – The heart and torso domains

$$\left\{\begin{array}{ll} -div(\boldsymbol{\sigma}\nabla u) = f\mathbf{1}_{\Omega_H} & \text{in } \Omega \\ \boldsymbol{\sigma}\nabla u.n_T = 0, & \text{on } \Gamma_{\text{ext}}. \end{array}\right. \tag{3}$$

2 Topological gradient method

We use two notations of the cost function to be minimized : $j(\Omega_H)$ and $\mathfrak{J}(u_{e,\Omega_H})$, where u_{e,Ω_H} is the solution to the system (1). The idea of topological asymptotic analysis is to measure the effect of a perturbation of the domain Ω_H on the cost function. For a small $\epsilon \geq 0$, let $\Omega_\epsilon := \Omega_H \setminus \overline{\theta_\epsilon}$ be the perturbed domain by the insertion of an inclusion $\theta_\epsilon = x_0 + \epsilon\theta$, where $x_0 \in \Omega_H$ and θ is a given, fixed and bounded domain of $\mathbb{R}^d$, containing the origine, whose boundary $\partial\omega$ is C^1. The topological sensitivity theory provides then an asymptotic expansion of the considered cost function when the size of ω_ϵ tends to zero. It takes the general form :

$$j(\Omega_\epsilon) - j(\Omega_H) = \rho(\epsilon)g(x_0) + o(\rho(\epsilon)),$$

where $\rho(\epsilon)$ is an explicit positive function going to zero with ϵ, and $g(x_0)$ is the topological gradient at point x_0. Then in order to minimize the criterion, one has to insert small inclusion at points where the topological gradient is the most negative. In our case, the source would be identified in the zones where the topological gradient is the most negative. $j(\Omega_\epsilon)$ would be a function minimizing the gap between the solution u_ϵ solution of the following problem and a given observed data.

$$\left\{\begin{array}{ll} -div(\boldsymbol{\sigma}\nabla u_\epsilon) = f_\epsilon\mathbf{1}_{\Omega_H} & \text{in } \Omega, \\ \boldsymbol{\sigma}\nabla u_\epsilon.n_T = 0, & \text{on } \Gamma_{\text{ext}}, \end{array}\right. \tag{4}$$

where

$$f_\epsilon = \left\{\begin{array}{ll} f_1 & \text{on } \theta_\epsilon \\ f_0 & \text{on } \Omega_\epsilon. \end{array}\right.$$

is the unknown source to be identified.

2.1 Variational formulation

The solution of the problem (4) is defined up to a constant, thus we define the suitable functional space by

$$V = \{v \in H^1(\Omega) \quad , \quad \int_{\Omega_H} v = 0\}$$

and the bilinear form A_ϵ and the linear form l_ϵ as

$$A_\epsilon(u_\epsilon, v) = \int_\Omega \boldsymbol{\sigma} \nabla u_\epsilon \nabla v \quad and \quad l_\epsilon(v) = \int_\Omega f_\epsilon v \quad \forall v \in V$$

Then the variational formulation of this problem reads such that

$$\int_\Omega \boldsymbol{\sigma} \nabla u_\epsilon \nabla v = \int_\Omega f_\epsilon v, \forall v \in V.$$

The solution u_ϵ is solution of $A_\epsilon(u_\epsilon, v) = l_\epsilon(v), \forall v \in V$. To determine the topological gradient we need to compute the adjoint solution of this problem.

2.2 Adjoint problem

We consider the direct solution u_ϵ satisfying $A_\epsilon(u_\epsilon, v) = l_\epsilon(v)$ and we define the lagrangian $L_\epsilon(u, p) = \mathfrak{J}(u) + A_\epsilon(u, p) - l_\epsilon(p)$, for every $u, p \in V$. One could check that if u_ϵ is solution of (4) we have

$$L_\epsilon(u_\epsilon, v) = \mathfrak{J}(u_\epsilon)$$

We denote $D_u L_\epsilon$ and $D_u \mathfrak{J}$ the derivative of L_ϵ and $\mathfrak{J}$ respectively, so

$$D_u L_\epsilon(u_\epsilon, v) = D_u \mathfrak{J}(u_\epsilon)$$

Then we define the abstract adjoint equation by

$$(D_u L_\epsilon, \psi) = 0, \forall \psi \in V$$

we have

$$(D_u \mathfrak{J}(u), \psi) + \int_\Omega \boldsymbol{\sigma} \nabla p \nabla \psi = 0$$

So

$$\int_\Omega \boldsymbol{\sigma} \nabla p \nabla \psi = -(D_u \mathfrak{J}(u), \psi)$$

Finally the adjoint solution p associated of the cost function $\mathfrak{J}$ is given by

$$\begin{cases} -div(\boldsymbol{\sigma} \nabla p) & = -D_u \mathfrak{J}(u) \ \text{ in } \ \Omega \\ \nabla p \cdot n_T & = 0 \qquad\qquad\ \ \text{ on } \ \Sigma, \end{cases} \tag{5}$$

We remarque that the computation time and memory space required by the state adjoint method are largely reasonable. In the next section we will derive the variation of the cost function j with respect to the insertion of a small subdomain ω_ϵ in the cardiac domain Ω_H. We begin our analysis by giving the main hypothesis 1, then the main result of this section is presented by Theorem 1. It concerns the topological asymptotic expansion of a cost function $\mathfrak{J}$.

2.3 Main result

Let us consider the following hypothesis :

hypothesis 1 *We assume That*
(i) $\mathfrak{J}$ is differentiable with respect to u, we denote $D\mathfrak{J}(u)$its derivative.
(ii) There exists a real number $\partial J(x_0)$ such that

$$\mathfrak{J}(u_\epsilon) - \mathfrak{J}(u_0) = D\mathfrak{J}(u_0)(u_\epsilon - u_0) + \epsilon^d |\omega_\epsilon| \partial J(x_0) + o(\epsilon^d)$$

(iii) $\|u_\epsilon - u\|^2_{L^2(\partial\Gamma_{ext})} = o(\epsilon^d)$
(iv) $\|\nabla(u_\epsilon - u)\|^2_{L^2(\partial\Gamma_{ext})} = o(\epsilon^d)$

The expression of the topological gradient for this problem is given by the following result :

Theorem 1 *Under the hypothesis above the cost function j has the following asymptotic expansion :*

$$j(\Omega_\epsilon) - j(\Omega_H) = \epsilon^d |\omega_\epsilon| \partial J(x_0) - \epsilon^d |\omega_\epsilon| (f_1 - f_0) p(x_0)$$

In other words, the topological gradient at x_0 is :

$$g(x_0) = \partial J(x_0) - (f_1 - f_0) p(x_0)$$

where p is the adjoint solution.

Proof 1 *We always seek to minimize the function $\mathfrak{J}$ defined above. We consider the lagrangian*

$$L_\epsilon(u, v) = \mathfrak{J}(u) + A_\epsilon(u, v) - l_\epsilon(v)$$

u_ϵ is solution of problem 4, then we have

$$j(\Omega_\epsilon) = L_\epsilon(u_\epsilon, v)$$

So the first variation of the cost function with respect to ϵ is given by

$$\begin{aligned} j(\Omega_\epsilon) - j(\Omega_H) &= L_\epsilon(u_\epsilon, v) - L_0(u_0, v) \\ &= \mathfrak{J}(u_\epsilon) - \mathfrak{J}(u_0) + A_\epsilon(u_\epsilon, v) - A_0(u_0, v) - l_\epsilon(v) + l_0(v) \end{aligned}$$

Then from the definition of A_ϵ and l_ϵ we have :

$$A_\epsilon(u_\epsilon, v) - A_0(u_0, v) = \int_\Omega \boldsymbol{\sigma} \nabla(u_\epsilon - u_0) \nabla v$$

$$l_\epsilon(v) - l_0(v) = \int_{\omega_\epsilon} (f_1 - f_0) v$$

Choosing $v = p$ the adjoint solution is solution of (5)

$$\int_\Omega \boldsymbol{\sigma} \nabla(u_\epsilon - u_0) \nabla p = -D\mathfrak{J}(u_0)(u_\epsilon - u_0)$$

Then we have

$$j(\Omega_\epsilon) - j(\Omega) = \mathfrak{J}(u_\epsilon) - \mathfrak{J}(u_0) - DJ(u_0)(u_\epsilon - u_0) - \int_{\omega_\epsilon} (f_1 - f_0) p$$

From the hypothesis we have

$$j(\Omega_\epsilon) - j(\Omega_H) = \epsilon^d |\omega_\epsilon| \partial J(x_0) - \epsilon^d |\omega_\epsilon| (f_1 - f_0) p(x_0)$$

So we have

$$j(\Omega_\epsilon) - j(\Omega_H) = \rho(\epsilon) g(x_0) + o(\rho(\epsilon))$$

where

$$g(x_0) = \partial J(x_0) - (f_1 - f_0) p(x_0)$$

where $\partial J(x_0)$ depends on the cost function. We will present in the next section some examples of the cost function and the associated $\partial J(x_0)$ term.

3 Numerical results

In this paragraph we aim to recover the source term with the help of the non-invasive observations on the external boundary of the torso. We use the bidomain model in order to create a source term based on reaction diffusion equation. We solve the electrostatic source identification problem at a given time step. The topological gradient method is implemented using the following algorithm :

- Solve the forward solution of the problem 4.
- Compute the adjoint solution of the problem 5.
- Compute the topological gradient g.
- Search for the minimum of the topological gradient.

In order to numerically test the topological gradient method, we consider a two cost functions $\mathfrak{J}_1(u) = \int_{\partial\Gamma_{ext}} |u - u_{obs}|^2 dx$ and $\mathfrak{J}_2(u) = \int_{\partial\Gamma_{ext}} |\nabla u - \nabla u_{obs}|^2 dx$, where u_{obs} is the observed data at the body surface Γ_{ext}. We tested this method for both cost functions in two different scenarios. The first case is for clustered source. The electrical source in this case is obtained by solving the bidomain equation with a single site stimuli until 4ms. The second case is for a distributed source, The electrical source in this case is the gradient of the transmembrane potential at 20 ms after a single site stimuli.

clustered source :

In figure 2 (a), we show the distribution of the extracellular potential in the heart domain after 4ms of a single site stimulation. The topological gradient distribution is shown in figure 2 (b) for the cost function $\mathfrak{J}_1$ and figure2 (b) for the cost function $\mathfrak{J}_2$. The green circle in figures 2 (b,c,e,f) denotes the position of the source at 4 ms and the red point is the source obtained using the topological gradient method. The source at time 4ms could be deduced from figure 2 (e), where we represent the distribution of $f = div(\boldsymbol{\sigma}_\mathbf{i} \nabla V_m)$. We distinguish two clustered sources. We remark that the electrical source is globally well localized. The two cost functions seems to capture one of the two sources at time 4ms.

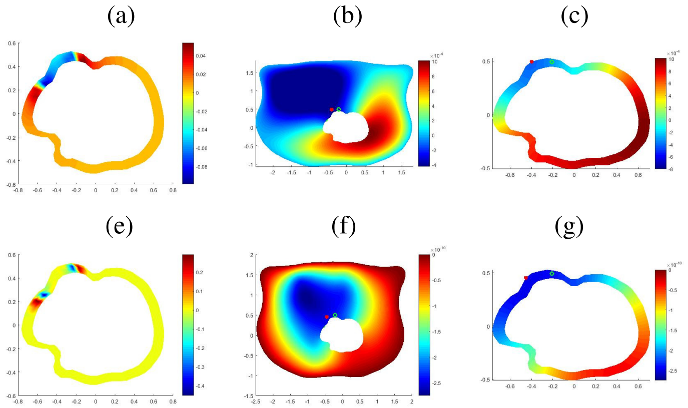

Figure 2 – (a) the solution u_e at 4 ms, (e) : the source. (b) (respectively,(c)) : the topological gradient for the cost function $\mathfrak{J}_1$ (respectively,$\mathfrak{J}_2$) in the heart thorax doamin. (f) (respectively,(g)) : The topological gradient for the cost function $\mathfrak{J}_1$ (respectively,$\mathfrak{J}_2$) in the heart doamin.

Distributed source

Here we test the capability of the method in localizing distributed sources. We run a simulation of a single site stimuli and we extract the data after 20 ms. In figure 3 (a), we show the distribution of the extracellular potential in the heart domain. The topological gradient distribution is shown in figure 3 (b) for the cost function $\mathfrak{J}_1$ and figure3 (b) for the cost function $\mathfrak{J}_2$. The green circle in figures 3 (b,c,e,f) denotes the position of the source at 20 ms and the red point is the source obtained using the topological gradient method. The source at time 20ms could be deduced from figure 3 (e). We distinguish two sources far from each other. We remark that the first cost function still provides an averaged position which is here very far from both real sources figure 3 (e). By the contrary, the second cost function still captures with a good accuracy one of the two sources at time 20 ms.

4 Conclusion

We presented a new approach for localizing electrical sources in the heart. This approach is based on the topological gradient method. We have tested this method on in silico data obtained by solving the bidomain problem. The numerical results show that the method is accurate when dealing with clustered sources. Our investigation shows that the considering the cost function $\mathfrak{J}_2(u) = \int_{\partial\Gamma_{ext}} |\nabla u - \nabla u_{obs}|^2$ is better than considering $\mathfrak{J}_1(u) = \int_{\partial\Gamma_{ext}} |u - u_{obs}|^2$. The first capture one of the two sources. The latter tries to find an averaged position. This works well when the source is clustred but when the sources are far from each other, the function $\mathfrak{J}_2(u)$ seems to localise the source that is the closest to the body surface. These preliminary results have been conducted in 2D simulations and have to be confirmed with much more testing with multiple stimuli and multiple sources for the 2D and the 3D cases. This would be the topic of our future investigations.

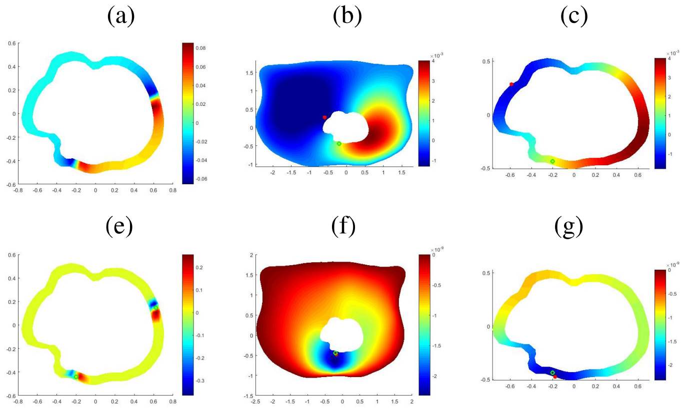

Figure 3 – (a) the solution u_e at 20 ms, (e) : the source. (b) (respectively,(c)) : the topological gradient for the cost function $\mathfrak{J}_1$ (respectively,$\mathfrak{J}_2$) in the heart thorax doamin. (f) (respectively,(g)) : The topological gradient for the cost function $\mathfrak{J}_1$ (respectively,$\mathfrak{J}_2$) in the heart doamin.

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