A spatio-temporal model for phenomena dynamics based on 2D Diffusion equations

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ABSTRACT. This study highlights the relevance of the dynamics of population in an environment in the prediction of phenomena. A hybrid model compatible with a 2D diffusion equation is proposed. It is based on the balanced method coupled with the models of the dynamics of the populations. The resulting equations, since they are a kind of conservation equations, are discretized using the finite volume method. This equation is strongly linked to a probabilistic diffusion coefficient which highlights the random moving of mobile entities. It also represents the influence of the neighbors of each site in the dynamics of mobile entities, within a closed environment. This approach is illustrated on the well-known SIR epidemiological model to produce a variant which consider the spatio-temporal aspect of the spread.

KEYWORDS: Population Dynamics, Prediction, Diffusion, Finite Volume, Random, Closed Environment

1. Introduction

The observation of the phenomena is a major research activity which deals with the prediction based on models and theories sometimes existing or created. Today, with the development of new technologies, migrations, wars, the mixing of populations, terrorism etc. The prediction becomes decision support tool. Given the complexity of this field, there are several approaches due to the random behavior of mobile entities in one hand and geographic constraints on the other. It would be therefore possible to circumvent these difficulties by considering for each phenomenon a theory or a model that adapts to it as well as possible.

In this study, we work in a closed environment divided into several distinct sites. In this environment the mobile entities move from one site to another in a random manner, thus changing the size of the population.

The most researches often put emphasis on the phenomenological meaning around a theme. The contrast between ideal and reality, the expected and lived brings us to think alike in prediction domain by studying a phenomenon in all its outlines with all its characteristics and make that the rendering of its study is reliable, its modes of emergence or manifestation can be elaborated in the form of mathematical equation. The conclusion is clear, most prediction models take much more into account the dynamics of the phenomenon to be studied without explicitly highlighting the impact of the spatio-temporal dynamics of the entities concerned by the prediction model. Hence we have got the following ideas

- Study the behavior of a phenomenon dynamically as a function of time and space in its evolution

- Take a phenomenon into an atomic division according to the spatial-temporal behaviors and to superpose afterwards to find the global behavior

- Find for each phenomenon a spatio-temporal behavior which adapts to an existing model or to propose a model

- Dissociate the dynamics of the individuals to that of a phenomenon then to associate the different results to predict a future behavior

- To write probabilistic time equations in order to derive at a model that can integrate the random aspect of a phenomenon and its spatio-temporal evolution.

Some authors (Jianhong Wu and P. van den Driessche [4]) thought in this way. Without pretending to control the manifestation of all the contours related to a phenomenon in the case of prediction, our contribution will be much more oriented towards the impact of the spatio-temporal dynamics of the mobile entities in the prediction models. [5] [4].

In the remainder of this paper the section 2 present the related works which inspired our point of view, a particular emphasis will be put on those that allowed us to build our model. The section 3 describes the approach used to build the equations underline the proposed model. The section 4 show an illustration on the extension of the SIR epidemiological model then the paper end with conclusions and perspectives.

2. Literature review.

2.1. Particle diffusion

The idea underlying this study consist to extend the diffusion equation applied to the analysis and the prediction of phenomenon. Starting from the mobile particle balanced method based on the diffusion of particles [10], the following 1D diffusion is build

$$\frac{\partial n(x,t)}{\partial t} = -\frac{\partial j_x(x,t)}{\partial x} \quad (1)$$

Where $\vec{j}(x,t)$ is the density of moving entities and n(x,t) the number of mobile entities per volume unit. For 2D and more the equation (1) takes the following form

$$\frac{\partial n(M,t)}{\partial t} + div\vec{j} = 0 \quad (2)$$

It is proved that these equations are valid in any geometry the divergence operator can change depending on the coordinate system adopted.

2.2. Population dynamics: parabolic partial differential equation

Jimmy Garnier in his thesis [5] stutied this family of equation

$$\frac{\partial u(t,x)}{\partial t} = D(u)(t,x) + f\big(x, u(t,x)\big) \qquad t > 0\,,\ x\ \in R \quad (3)$$

These equations are useful in many fields such as combustion chemistry, biology or ecology. These equations are generally used to modelize the evolution of entities that interact each other and moved. f population dynamics or population genetics[7]. The quantity u (t, x) represents the population density at time t and at position x. The reaction term f (x, u) corresponds to the growth rate of the population. This term of reaction depends on the one hand on the density u and on the other hand with the medium in which the population evolves through the variable of space x.

In this large set, we will focus mainly on a single type of reaction-dispersion equation where the dispersion operator D is a second-order elliptic differential operator. [5]

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f\big(x, u(t,x)\big) \quad (4)$$

A particular attention is focused on this equation because it looks like the diffusion equation obtained from the fick law which state that the flow due to the random movement is approximately proportional to the gradient in the number of individuals.

In the same way Jianhong Wu [6] starting from the basic concepts to developed a model for the spatial spread of diseases involving hosts in random displacement during certain stages of the progression of the disease. He got a diffusion model based on the conservation laws and Fick's law. This model was applied to the study of two cases, namely the spread of rabies in continental Europe during the period 1945-1985 and the rates of spread of West Nile virus in North America.

The same approach is used in 2D closed environment to avoid solving equation (1) which has two unknowns. To underline the randomness aspect in displacement, the probabilistic diffusion coefficient will be built as Wu [6]. This coefficient will also take into account the dimensional aspect of the quantities used

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3. Methodology

3.1. Basis of the model

Starting from the diffusion of the particles in 2 dimension assuming that the particles move along x and y coordinates, we obtain the fallowing 2D conservation equation

$$\frac{\partial n(M,t)}{\partial t} + div\vec{j} = 0 \quad (2)$$

In the following it is necessary:

- Find an explicit form of equations (1) and (2) as a function of time and space
- Find the equivalence of the operator div (j (M, t)) as a function of density of mobile entities n (M, t)

Fick's law mentioned in [4] states that the flow due to the random movement is approximately proportional to the gradient in the number of individuals like

$$j = -D\frac{\partial n(t,x)}{\partial x}$$

To have a diffusion equation of the form

$\frac{\partial n(x,t)}{\partial t} = D\Delta n(x,t)$ (4) Where D is the diffusion coefficient and Δ is the Laplacian operator.

We have exploited the parabolic equation below ta ken from [5]

$\frac{\partial u(t,x)}{\partial t} = D(u)(t,x) + f\big(x, u(t,x)\big)$ (3) $\quad t > 0\,,\ x \in R$. In the difusion form as:

$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f\big(x, u(t,x)\big)$ (4)

According to our analysis, it combines particle scattering and Fick's law, notwithstanding a residual coefficient D (diffusion coefficient). This went well to the form of equation sought. To continue in our positioning we made approximations on equation (4) as follows:

- We first neglect the creation factor f (x, u (t, x)) taking into account the time difference between t and t + dt which will not be at the scale of a duration that can hold significantly account for the death or birth of a new mobile entity.
- Add a coefficient D in front of the second-order elliptic differential operator to take into account the randomness of the displacement of moving entities from one site to another on the one hand and the homogeneity of the dimensional equation on the other hand.
- The coefficient D can be constant or follow a law of variation according to the complexity related to the motions inter - site of the mobile entities.

In case of 2 dimension we got the following equation

$\frac{\partial u(t,x)}{\partial t} = D(\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial^2 u(t,x)}{\partial y^2})$ (5)

Thus the whole difficulty of this modeling will reside on our capacity to give a form adapted to the diffusion coefficient D. Seen in this angle the diffusion coefficient D will make our model adaptive.

3.2. Model development

For a first approach we will consider our closed environment as a homogeneous site distribution

The complexity of our approach takes us to a discrete solution seen the impossibility to have an analytical solution because of the randomness of the displacement of the moving entities. For that we chose the method of the finished volumes [9].

For the diffusion coefficient D we modeled it as a transition matrix that materializes the contribution of a site x to a site y during the diffusion. So we have made assumptions that lead us to formulate the coefficient D in the following ways

- A mobile entity in a site can decide to move or not
- the probability of moving from one site to another depends on the number of neighbor's sites

- $D_{x,y} = \omega P_{x,y}$ $with$ $\omega \approx \frac{\Delta x^2}{\Delta t}$

- ω depend on the configuration of the problem (as a function of the characteristic speed of the movement of the mobile entities).

- $P_{x,y}$ the probability of leaving a site x for a site y

After applying the finite volume method we obtain the following numerical scheme

$$u_{i,j,k}^{n+1} = \frac{\Delta t D_{l,k}}{\Delta x^2}\left(u_{i+1,j,x}^{n}\delta_{l,x} + u_{i-1,j,y}^{n}\delta_{l,y}\right) + \frac{\Delta t D_{l,k}}{\Delta y^2}\left(u_{i,j+1,z}^{n}\delta_{l,z} + u_{i,j-1,t}^{n}\delta_{l,t}\right) + \left(1 - 2\left(\frac{\Delta t D_{l,k}}{\Delta x^2} + \frac{\Delta t D_{l,k}}{\Delta y^2}\right)\right)u_{i,j,k}^{n}\delta_{l,k} \qquad (6)$$

Where $\delta_{i,j}$ are the kronecker's symbols and $u_{i,j,k}^{n}$ the mobile entity density

It is the general formulation of the model. We take into account the boundary conditions according to the considered space geometry. In this formula n represents the temporal discretization index i and j the spatial discretization indices along the x axis and the y axis. k represents the number assigned to a site used as an index in the diffusion matrix

4. Illustration

This section, is the ways of presenting our ideas of the taking into account the spatial-temporal factors to bring a corrective term to a familiar model of prediction and to change its original form. Suppose a homogeneous population and each individual of the population can be identified by its position in a site within our enclosed environment. Take the case of a disease that acts on mobile entities and whose dynamics are modeled by the epidemiological model SIR (Susceptible, Infective, and Recovered) and combined with our probabilistic diffusion modeling approach. Given our assumptions, our modeling should lead us to ordinary differential equations at first if we do not take into account the inter-site migration described above in our modeling. Then we will have another system of differential equations that highlights the impact of the inter-site migration. That is the goal of this example modify the equations of an ordinary system by taking into account the inter-site migration in the equations of the model. In the following, we assume that it is the same disease that occurs and spreads across all the different sites involved in trade For this purpose, let α, β and γ be respectively the rates of infection, cure and return to the susceptible condition of the individuals of a site, and assuming that we have P sites, we obtain in a first time a set of 3P ordinary differential equations describing the dynamics of infection within the population of a site i. Then, in a second step, a set of 3P complex differential equations describing the dynamics of the infection combined with the inter-site dynamic of the populations

- Characteristic equation system for the 3P differential equations without taking into account the dynamics between sites

$$(7)\begin{cases}\frac{dS(t)}{dt} = \gamma R - \alpha S \\ \frac{dI(t)}{dt} = \alpha S - \beta I \\ \frac{dR(t)}{dt} = \beta I - \gamma R\end{cases}$$

- Characteristic equation system for 3P differential equations taking into account inter-site dynamics

$$(8)\begin{cases}\frac{\partial S(t)}{\partial t} = \gamma R - \alpha S + D\Delta S \\ \frac{\partial I(t)}{\partial t} = \alpha S - \beta I + D\Delta I \\ \frac{\partial R(t)}{\partial t} = \beta I - \gamma R + D\Delta R\end{cases}$$

After having this new equation family we discretize with the finite volumes method according to the pre-established model. We make a small simulation with a dataset using a code written in python language to have a result that shows the modification brought by the spatio-temporal consideration. The simulation takes place in a closed environment with 4 sites.

The results are presented in the annexes in the form of a histogram showing the differences between the results based on the diffusive SIR model and those of the native SIR model.

These results show us that when the diffusion rate is greater than the contamination coefficients in the natural SIR model, the rate of contamination decreases because the entities disappear from a site even before the disease has time to spread.

In conclusion we can say that it will be possible to take into account the spatio-temporal aspect in the prediction of the phenomena as a function of the diffusion speed because the more important it is the more it influences the dynamic of the studied phenomenon.

5. Conclusion

The purpose of this article was to provide a corrective factor in the prediction models, by considering the spatio- temporal impact within the dynamics of a prediction model. For this purpose a particular attention was focused through several models of prediction dealing with mobile entities. Upon certain clearly defined hypothesis, we designed a hybrid model based on diffusion equations. This approach was illustrated by modifying a naturel SIR model to have another hybrid equation system. The validation is done through a simulation with a dataset. The results show that the aspect of spatio-temporal dynamics modifies the behavior of the native SIR model.

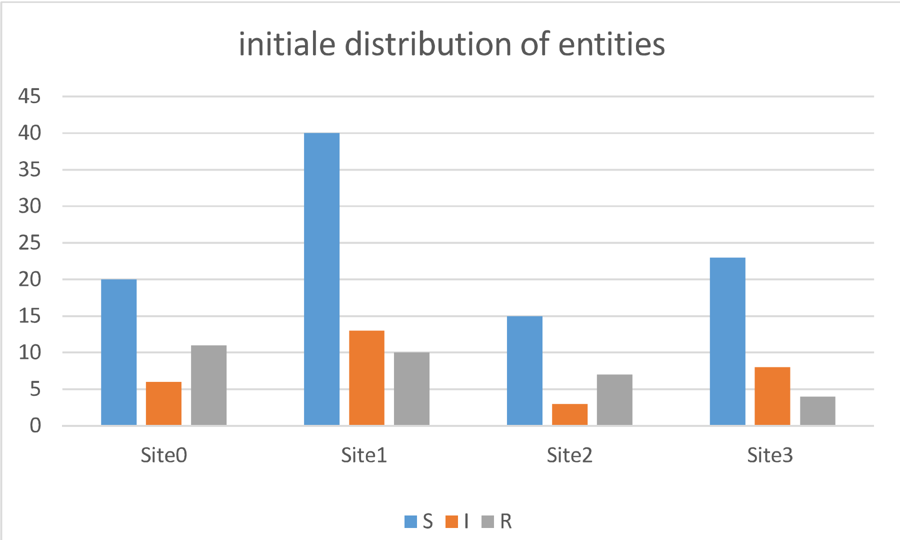
The further work intend to bring the model closer to the reality. In this regard the following ideas will be develop

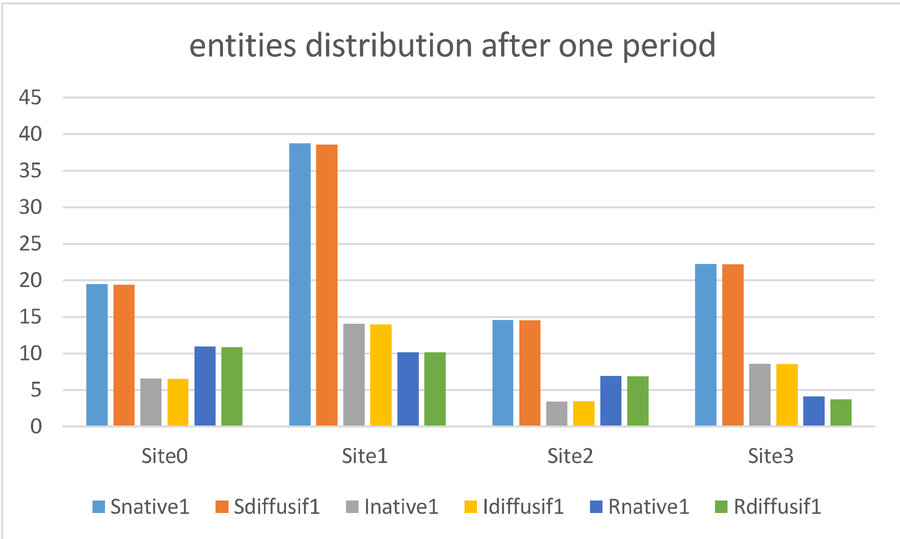
- The discretization will be done in irregular mesh
- The formalization of the method of building the probality of moving
- propose the intervals of diffusion speed depending on the nature of the problem addressed

6 Bibliographie

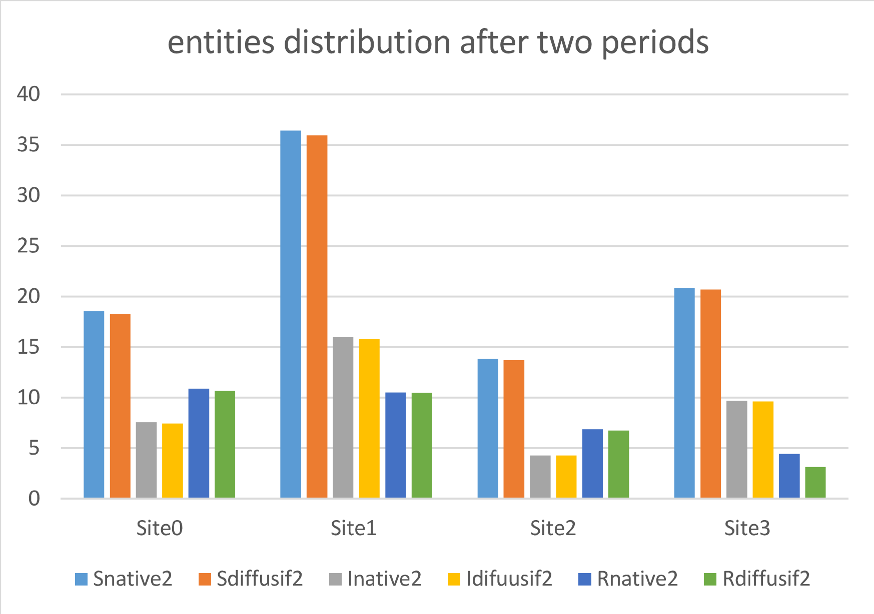
[1] Allard, A. (2008). *Modélisation Mathématique en épidémiologie par les réseaux de contacts : Introduction de l'Hétérogénéité dans la Transmissibilité*, M.Sc. Theses. Université de Laval, Quebec
[2]. BASILEU, C. (2011*). Modélisation structurelle des réseaux sociaux : Application à un système d'aide à la décision en cas de crise sanitaire*. PhD Thesis, Université Lyon 1
[3]. Djamila Moulay. *Modélisation et analyse mathématique de systèmes dynamiques en épidémiologie. Application au cas du Chikungunya. Mathématiques* [math]. PhD Thesis Université du Havre, 2011. Français. <tel-00633827 >
[4]. Driessche, P. v. (2008). *Spatial Structure: Patch Models*. Springer Berlin Heidelberg, Lecture Notes in Mathematics Volume 1945, 2008, pp 179189
[5]. Jimmy Garnier. *Analyse mathématique de modèles de dynamique des populations : équations aux dérivées partielles paraboliques et équations intégro-différentielles*. Equations aux dérivées partielles [math.AP]. A ix-Marseille Université, 2012. Français. <tel-00755296 >
[6] Julien Arino. (2008). *Diseases in metapopulations. Centre for Disease Modelling*, Preprint 200804, University of York.
[7] N S Kolmogorov, N Petrovsky et I G Piskunov : *Étude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*. Bull. Univ. Moscou, Sér A, 1:1–26, 1937.
[9] Robert Eymard, Thierry Gallouet and Raphaele Herbin *Finite Volume Methods.* January 2003 This manuscript is an update of the preprint n° 97-19 du LATP, UMR 6632, Marseille, September 1997whichappeared in Handbook of Numerical Analysis, P.G. Ciarlet, J.L. Lions eds, vol 7, pp 713-1020
[10] Salamito, BernardCardini, Stéphane Jurine, Damien *physique tout en un* Editeur: Dunod Publication: 2013 pages: 1124 ISBN: 978-2-10-060077-9

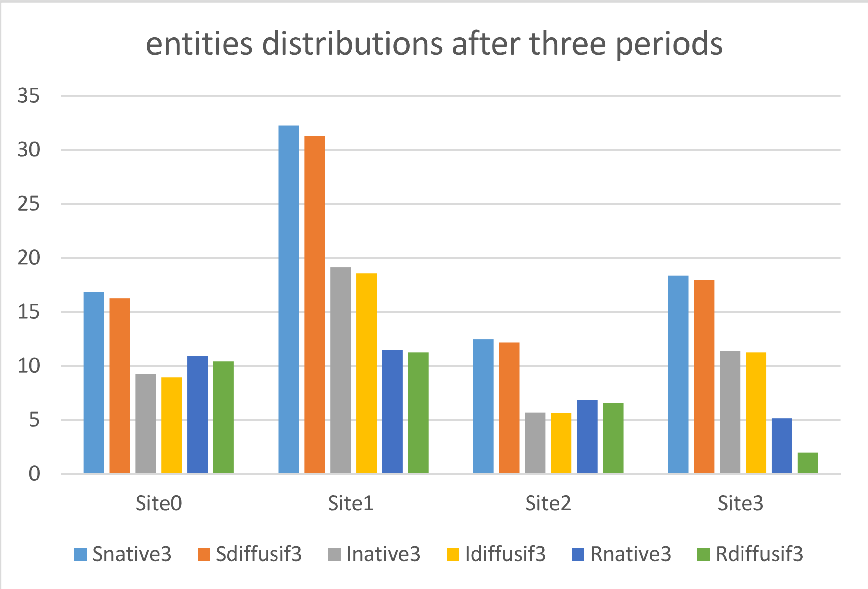
Annex





Snative = S in native SIR, **Sdiffusif** = S with diffusif model of SIR. Then the same thing is made with I and R to compare data in the same histogram


entities distribution after two periods
40
35
30
25
20
15
10
5
0
Site0
Site1
Site2
Site3
Snative2
Sdiffusif2
Inative2
Idifuusif2
Rnative2
Rdiffusif2


entities distributions after three periods
35
30
25
20
15
10
5
0
Site0
Site1
Site2
Site3
Snative3
Sdiffusif3
Inative3
Idiffusif3
Rnative3
Rdiffusif3