# Coupling Discontinuous Galerkin method and integral representation for solving Maxwell's system

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**ABSTRACT.** we present a mathematical and numerical study of the three-dimensional time-harmonic Maxwell equations solved by a discontinuous Galerkin method coupled with an integral representation. This study was completed by some numerical tests to justify the effectiveness of the proposed approach.

**RÉSUMÉ.** nous présentons une étude mathématique et numérique pour la résolution des équations de Maxwell tridimensionnelles en régime-harmonique, par une méthode de type Galerkin discontinu couplée à une représentation intégrale. Cette étude a été complétée par des tests numériques pour justifier l'efficacité de l'approche proposée.

**KEYWORDS :** Maxwell equations, time-harmonic, discontinuous Galerkin method, integral representation, fictitious domain

**MOTS-CLÉS :** Équations de Maxwell, régime-harmonique, méthode de Galerkin discontinu, représentation intégrale, domaine fictif

## 1. Introduction

The propagation of electromagnetic waves is a physical phenomenon that describes the analysis of an emitted wave, this phenomenon is described by mathematical equations. In this work, the electromagnetic wave propagation equation will result in Maxwell's equations.

What's interesting for Maxwell's equations is that the domain of validity extends to a wide variety of waves: radar, TV, radio, ... and even in radiation fields as varied: Ultraviolet, X-rays, infra-red, gamma, etc.

Various methods have been developed for numerical resolution of Maxwell's equations, however, it seems that no method is predominant if we take into account the BF–MF–HF domains we are interested in BF–MF domains. Our work in this paper is devoted to the resolution of three-dimensional time-harmonic Maxwell's equations by the discontinuous galerkin method coupled to an integral representation.

## 2. Maxwell's problem

We are interested in this paper to the solutions of the time-harmonic Maxwell's equations in the presence of an obstacle $D$, which are particular solutions and which shall check the following system:

$$\left\{ \begin{array}{rcll} \nabla\times\ E + i\omega\mu\, H & = & J, & \text{in} \quad \mathbb{R}^n\backslash\overline{D}, \\ \nabla\times\ H - i\omega\varepsilon\, E & = & 0, & \text{in} \quad \mathbb{R}^n\backslash\overline{D}, \end{array} \right. \tag{1}$$

where $E$ and $H$ are respectively the electric and magnetic fields. The parameters $\varepsilon$ is the relative dielectric permittivity, $\mu$ is the relative magnetic permeability and $\omega$ is the pulsation.

So the perfect conductor condition will be considered on boundary $\Gamma_m$ define here:

$$\left\{ \begin{array}{rcl} E\ \times n & = & 0, \\ H\ .\, n & = & 0. \end{array} \right.$$

This problem is posed on an initially infinite domain; the idea here is to limit our domain by a fictitious boundary we will note it $\Gamma_a$.

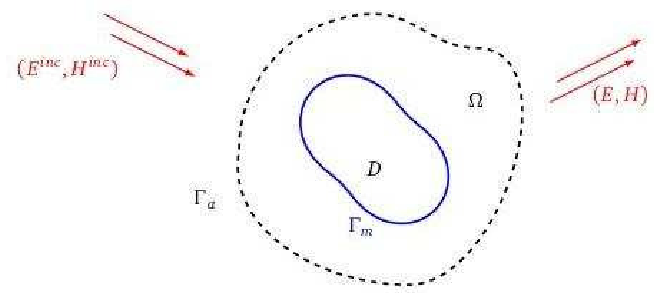


**Figure 1.** *Diffraction of an electromagnetic wave in the presence of an obstacle $D$ where its boundary is noted $\Gamma_m$*

We consider on this boundary an exact condition in the form of an integral representation defined by:

$$n \times E \,+\, n \times (n \times H) \;=\; n \times\, \mathfrak{R}(E) \,+\, n \times (n \times\, \mathfrak{R}(H)) \ ,$$

where $\mathfrak{R}(E)$ and $\mathfrak{R}(H)$ are respectively the values of $E$ and $H$ on $\Gamma_a$ expressed as a function of $E$ and $H$ in $\Gamma_m$ defined using the integral representation by the Stratton-Schu formulas [3, 4] given by:

$$\mathfrak{R}(E) = \mathcal{L}\, g \,-\, \mathcal{K}\, f \quad \text{and} \quad \mathfrak{R}(H) = \mathcal{L}\, f \,+\, \mathcal{K}\, g \ ,$$

where $f = n \times E$ , $g = -n \times H$ and for the fundamental solution of the Helmholtz problem (the Green function $G$):

$$\Big(\mathcal{G}\, u\Big)(x) \;=\; \int_\Gamma G(x,y)\; u(y)\; \mathrm{d}y \ , \quad \mathcal{L}\, u \;=\; \frac{1}{i\,k}\, \nabla \times \nabla \times \mathcal{G}\, u \quad \text{and} \quad \mathcal{K}\, u \;=\; \nabla \times \mathcal{G}\, u$$

For simplicity we assume that $J = 0$. At this phase, we then come back to a problem:

$$\left\{\begin{array}{lcllc} \text{Find } E, H \in H(\nabla\times,\Omega) & , & \text{such as:} & & \\ i\omega\varepsilon\, E - \nabla \times H & = & 0 & \text{in} & \Omega \\ i\omega\mu\, H + \nabla \times E & = & 0 & \text{in} & \Omega \\ n \times E & = & -n \times E^{inc} & \text{on} & \Gamma_m \\ n \times E - n \times (n \times H) & = & n \times \mathfrak{R}(E) - n \times (n \times \mathfrak{R}(H)) & \text{on} & \Gamma_a \end{array}\right. \qquad (2)$$

where $H(\nabla\times,\Omega) = \{v \in L^2(\Omega)^3 : \nabla \times v \in L^2(\Omega)^3\}$ and:

$$E^{inc} = \begin{bmatrix} E_1^{inc} \\ E_2^{inc} \\ E_3^{inc} \end{bmatrix} \quad , \; E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \quad , \; H = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix} \quad \text{et} \;\; n = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

In the vector field $W$ such that $W = \begin{bmatrix} E \\ H \end{bmatrix}$, the problem (2) will be written in this matricial form:

$$\left\{\begin{array}{ll} i\omega Q W + G_x \partial_x W + G_y \partial_y W + G_z \partial_z W & = 0 \;\; \text{on } \Omega \\ (M_{\Gamma_m} - G_n)\; (W + W^{inc}) & = 0 \;\; \text{in } \Gamma_m \\ (M_{\Gamma_a} - G_n)\; (W - \mathfrak{R}(W)) & = 0 \;\; \text{in } \Gamma_a . \end{array}\right. \qquad (3)$$

Where $Q = \begin{bmatrix} \varepsilon\, I_3 & 0_{3\times3} \\ 0_{3\times3} & \mu\, I_3 \end{bmatrix}$

and $\mathfrak{R}(W) = \begin{bmatrix} \mathfrak{R}(E) \\ \mathfrak{R}(H) \end{bmatrix}$ such that: $\Big(\mathfrak{R}(W)\Big)(x) = \displaystyle\int_{\Gamma_m} K(x,y) W(y) \partial\sigma_y$ where $K : \mathbb{R}^6 \times \mathbb{R}^6 \to M_6(\mathbb{C})$ is a Green kernel.

In fact, denoting by $(e_x, e_y, e_z)$ the canonical basis of $\mathbb{R}^3$, the matrices $G_l$ for $k \in \{x, y, z\}$ are defined by:

$G_k = \begin{bmatrix} 0_{3\times3} & N_{e_k} \\ N_{e_k}^t & 0_{3\times3} \end{bmatrix}$ where for $l \in \{1,2,3\}$ a vector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $N_v = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix}$

Furthermore, $G_n = G_x\, n_1 + G_y\, n_2 + G_z\, n_3$.

$G_n^+$ and $G_n^-$ denote the positive and negative parts of $G_n$ [1]. We also define $|G_n| =$

1. If $P\Lambda P^{-1}$ is the natural factorization of $G_n$ then $G_n^{\pm} = P\Lambda^{\pm}P^{-1}$ where $\Lambda^+$ (resp. $\Lambda^-$) includes only positive eigenvalues (resp. negative).

$G_n^+ - G_n^-$. The matrices $M_{\Gamma_m}$ et $M_{\Gamma_a}$, are then defined by:

$$M_{\Gamma_m} = \begin{bmatrix} 0_{3\times 3} & N_n \\ -N_n^t & 0_{3\times 3} \end{bmatrix} \quad \text{and} \quad M_{\Gamma_a} = |G_n|$$

## 3. Variational formulation of the problem and discretization

We decompose the domain $\Omega$ in tetrahedral elements, we denote by $\tau_h$ the set of elements $K_i$.
For all $K_i \in \tau_h$, we define the functional space

$$V_h = \{W \in [L^2(\Omega)]^6 \; ; \; W_{|K_i} = W_i \in P_p(K)\}$$

By a development similar to that adopted by Ern and Guermond [1, 2] and adding the terms of the integral representation, the variational formulation of the problem (3) consists of:
$\forall \; V \in V_h \times V_h$, $K_i$ an element of $\tau_h$ obtained:
Find $W_i = (E_i, H_i) \in V_h \times V_h$ such as:

$$\begin{aligned}
\int_{K_i} (i\omega Q W_i)^t\, \overline{V} dx \quad & - \quad \int_{K_i} W_i^t \, ( \sum_{l\in\{x,y,z\}} G_l \partial_l \overline{V}) dx \\
& + \quad \int_{F\in\Gamma_i^0} \left[ (I_{FK_i} S_F \, [\![W_i]\!])^t \, \overline{V} + (I_{FK_i} G_{n_F} \, \{W_i\})^t \, \overline{V} \right] \partial\sigma \\
& + \quad \int_{F\in\Gamma_i^a} (\frac{1}{2}(M_{F,K_i} + I_{FK_i} G_{n_F}) W_i)^t \overline{V} \partial\sigma \\
& - \quad \int_{F\in\Gamma_i^a} (\frac{1}{2}(M_{F,K_i} - I_{FK_i} G_{n_F}) \Re(W_i))^t \overline{V} \partial\sigma \\
& + \quad \int_{F\in\Gamma_i^m} (\frac{1}{2}(M_{F,K_i} + I_{FK_i} G_{n_F}) W_i)^t \overline{V} \partial\sigma \\
& = \quad \int_{F\in\Gamma_i^m} (\frac{1}{2}(M_{F,K_i} - I_{FK_i} G_{n_F}) W_i^{inc})^t \overline{V} \partial\sigma
\end{aligned}$$

where: $\Gamma_i^0 = \bigcup_{K_j\in\tau_h} \overline{K_i} \cap \overline{K_j}$ , $\Gamma_i^m = \bigcup_{K_i\in\tau_h} \overline{K_i} \cap \Gamma_m$ and $\Gamma_i^a = \bigcup_{K_i\in\tau_h} \overline{K_i} \cap \Gamma_a$.
$I_{FK}$ represents the incidence matrix between facing surfaces and elements whose entries are given by:

$$I_{FK} = \begin{cases} 1 & \text{if } F \in K \text{ and orientations of } n_F \text{ and } n_K \text{ are match,} \\ -1 & \text{if } F \in K \text{ and orientations of } n_F \text{ and } n_K \text{ do not match,} \\ 0 & \text{if the face } F \text{ does not belong to the element } K. \end{cases}$$

where: $n_F$ is the unitary normal associated to the oriented face $F$ and $n_K$ is the unitary normal associated to the cell $K$.
We also define respectively the jump and average of a vector $V$ to $V_h \times V_h$ on the face $F$ shared between two elements $K$ and $\widetilde{K}$

$$[\![V]\!] = I_{FK} \, V_{|K} + I_{F\widetilde{K}} \, V_{|\widetilde{K}} \qquad \text{and} \qquad \{V\} = \frac{1}{2}\left(V_{|K} + V_{|\widetilde{K}}\right)$$

The matrices $S_F$ and $M_{F,K}$ are defined following the choice of numerical fluxes:

### 3.1. Centered flux

In this case, $S_F = 0$ and $M_{F,K} = \begin{cases} I_{FK} \begin{bmatrix} 0_{3\times3} & N_{n_F} \\ -N^t_{n_F} & 0_{3\times3} \end{bmatrix} & \text{if} \quad F \in \Gamma^m. \\ |G_{n_F}| \quad \text{if} \quad F \in \Gamma^a. \end{cases}$

### 3.2. Upwind flux

In this case, $S_F = \begin{bmatrix} \alpha^E_F N_{n_F} N^t_{n_F} & 0_{3\times3} \\ 0_{3\times3} & \alpha^H_F N^t_{n_F} N_{n_F} \end{bmatrix}$ and

$$M_{F,K} = \begin{cases} \begin{bmatrix} \eta_F N_{n_F} N^t_{n_F} & I_{FK} N_{n_F} \\ -I_{FK} N^t_{n_F} & 0_{3\times3} \end{bmatrix} & \text{if} \quad F \in \Gamma^m. \\ |G_{n_F}| \quad \text{if} \quad F \in \Gamma^a, \end{cases}$$

for a homogeneous medium, $\eta_F = \alpha^E_F = \alpha^H_F = \frac{1}{2}$

## 4. Linear system of the problem

We will treat the variational formulation term by term we can reduce our formulation in vectorial form

$$\left[D^1_i - D^2_i + D^{\Gamma_0}_i + \delta_{F^m_i}\, D^{\Gamma_m}_i + \delta_{F^a_i}\, D^{\Gamma_a}_i\right] W_i + \sum_{j\in V_i} E_{ij}\, W_j$$
$$+\delta_{F^a_i} \sum_{j:K_j\cap\Gamma_m \neq \varnothing} C_{ij}\, W_j = \delta_{F^m_i}\, B^{inc}_i$$

where: $D^1_i = i\omega \left(\Phi_i \otimes Q\right)$ , $D^2_i = \sum_{l=1}^{3} \left(\Phi^l_i \otimes G_l\right)$ ,

$D^{\Gamma^m}_i = \left(\Psi_{F^m_i} \otimes \left[\frac{1}{2}(M_{F,K_i} + I_{FK_i} G_{n_F})\right]\right)$ ,

$D^{\Gamma^a}_i = \left(\Psi_{F^a_i} \otimes \left[\frac{1}{2}(M_{F,K_i} + I_{FK_i} G_{n_F})\right]\right)$ ,

$D^{\Gamma^0}_i = \left(\Psi_i \otimes \left[I_{FK_i}(S_F\, I_{FK_i} + \frac{1}{2} G_{n_F})\right]\right)$ ,

$E_{ij} = \sum_{j\in V_i} \left(\Psi_{ij} \otimes \left[I_{FK_i}(S_F\, I_{FK_j} + \frac{1}{2} G_{n_F})\right]\right)$ ,

$C_{ij} = \frac{1}{2} \left(\Psi^{\Gamma^a_i} \otimes I_6\right)\ \widetilde{K}_{ij}\ \left(\Psi^{\Gamma^m_j} \otimes I_6\right)$ ,

$B^{inc}_i = Z_i W^{inc}_i = \left(\Psi_{\Gamma^m_i} \otimes \left[\frac{1}{2}(M_{F,K_i} - I_{FK_i} G_{n_F})\right]\right) W^{inc}_i$ ,

$F_{ij} = \overline{K_i} \cap \overline{K_j}$ , $F^m_i = \overline{K_i} \cap \Gamma_m$ , $F^a_i = \overline{K_i} \cap \Gamma_a$ ,

$V_i$ : the set of indices of neighboring elements of $K_i$ ,

$\delta_{F^a_i} = \begin{cases} 1 & \text{if} \quad \Gamma_a \cap K_i = F^a_i \\ 0 & \text{if} \quad \Gamma_a \cap K_i = \varnothing \end{cases}$ and $\delta_{F^m_i} = \begin{cases} 1 & \text{if} \quad \Gamma_m \cap K_i = F^m_i \\ 0 & \text{if} \quad \Gamma_m \cap K_i = \varnothing \end{cases}$

we can reduce our problem as a linear system:

$$(A - C)\, X = B$$

– $A$ is the square matrix of size:

$$N = 6 \times \underbrace{\text{Number of degrees of freedom}}_{d_i} \times \underbrace{\text{Number of cells}}_{N_c}$$

this matrix is a sparse matrix defined by block size $(6\ d_i \times 6\ d_i)$ such as:

- For $i = 1, \ldots, N_c$:

$$A(i,i) = D_i^1 - D_i^2 + D_i^{\Gamma^0} \times \delta_{ij} + D_i^{\Gamma^m} \times \delta_{\Gamma_i^m} + D_i^{\Gamma^a} \times \delta_{\Gamma_i^a}$$

- For $i, j = 1, \ldots, N_c$:

$$A(j,i) = E_{ij} \times \delta_{ij}$$

with:

$$\delta_{ij} = \begin{cases} 0 & \text{if} \quad K_i \cap K_j = \varnothing \\ 1 & \text{else} \end{cases}$$

– $C$ is a square matrix of the same size as $A$, defined by block size 6 $d_i \times 6\ d_j$ such as:

- For $i, j = 1, \ldots, N_c$:

$$C(i,j) = -C_{ij} \times \delta_{\Gamma_i^a} \times \delta_{\Gamma_j^m}$$

where:

$$\delta_{\Gamma_j^m} = \begin{cases} 0 & \text{if} \quad K_j \cap \Gamma_a = \varnothing \\ 1 & \text{else} \end{cases}$$

– $X$ is the vector of size $N$, Where its components are the unknowns of our problem.

– $B$ is the vector of size $N$ such as: $B(i) = B_i^{inc} \times \delta_{F_i^m}$

## 5. Numerical results

Following the mathematical study of the resolution of the Maxwell equations in unbounded domain by a method of type coupled with an integral representation (DG+IR), we present a sample of the numerical results.

We will give some numerical results by making the comparison between the approximate solution and the exact solution.

| Mesh | #M1 | #M2 | #M3 |
|---|---|---|---|
| Distance between $\Gamma_m$ and $\Gamma_a$ | 0.2 | 0.4 | 0.6 |
| $h_{max}$ | 0.1 | 0.1 | 0.1 |
| Number of elements | 204222 | 476454 | 830879 |
| Relative error (DG) | $0.467 \times 10^{-1}$ | $0.288 \times 10^{-1}$ | $0.286 \times 10^{-1}$ |
| Relative error (DG+IR) | $0.843 \times 10^{-2}$ | $0.883 \times 10^{-2}$ | $0.909 \times 10^{-2}$ |

**Table 1.** *Variation of external radius, k=5*

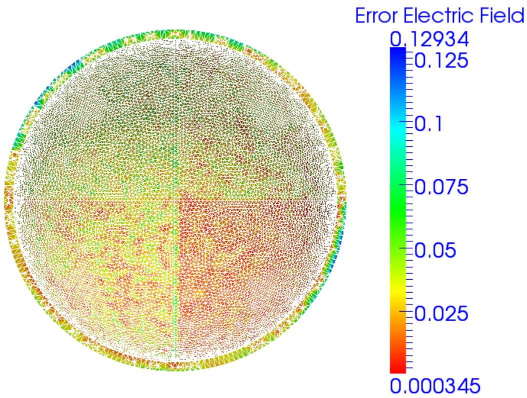


**Figure 2.** *Meshing of the volume between a first sphere of radius* $R = 1$ *and a second sphere of radius* $R = 1.06$*. A mesh size* $h = 0.07$*.*

## 5.1. Performance of methods with centered flux & upwind flux

The comparison results between the two methods DG+IR and DG are illustrated in the form of the relative error between the exact solution and the approximate solution either using a centered flux (see also figure (3)) or an upwind flux (see also figure (4)).

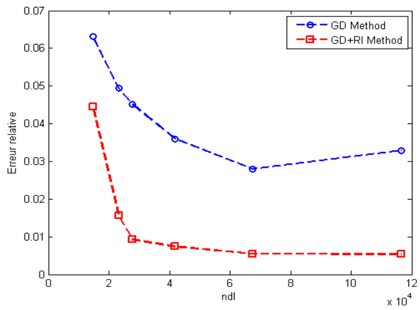


**Figure 3.** *Electric Field Error according to degree of freedom: Centered flux*

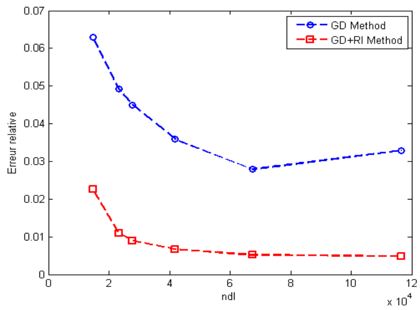


**Figure 4.** *Electric Field Error according to degree of freedom: Upwind flux*

A good improvement of the convergence is observed by using the DG method coupled to an integral representation using either a centered flux or an upwind flux.

## 5.2. Error depending on the size of the domain of study

We are interested in the case where the discretization step $h$ and the waves number $k = 10$ are fixed and by varying the distance delimited between the boundary of the obstacle $\Gamma_m$ and the artificial boundary $\Gamma_a$ by keeping a choice of wavelength equal to $20h$.

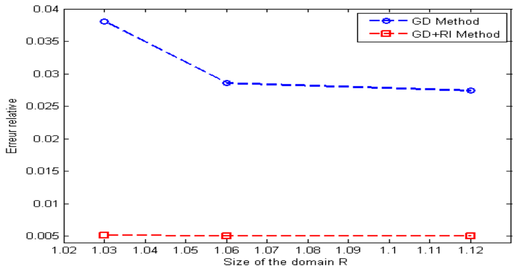


**Figure 5.** *Error according the size of the domain $R$.*

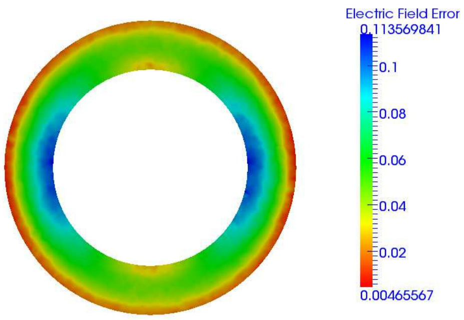


**Figure 6.** *Electric Field Error by the DG method.*

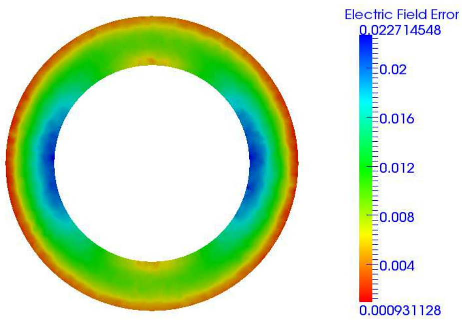


**Figure 7.** *Electric Field Error by the DG+IR method.*

It is clear that the results obtained by the DG+IR method are better, which shows that the coupling method is the most efficient. They show an show an improvement in accuracy, especially when the fictitious border is close to the boundary of the obstacle.

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