Identification of cracks in thermoelasticity

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RÉSUMÉ. Nous nous intéressons au problème inverse géométriques de l'identification de fissures en thermoélasticité. Les données considérées sont des mesures sur la frontière du déplacement et de la température. Nous étudions la stabilité de ce problème en nous limitant à l'identification d'une fissure émergente droite en milieu thermoélastique statique. Dans notre approche numérique pour identifier ces fissures, nous avons décomposé le processus en deux étapes; on identifie la direction puis la longueur de la fissure. Nous étudions aussi l'unicité de la solution du problème d'identification des fissures dans un cas plus général; nous prouvons l'identifiabilité d'une fissure modélisée par une courbe lisse via un modèle thermoélastique non statique.

ABSTRACT. We are interested on the geometric inverse problem of identification of cavities and cracks in thermoelasticity. The considered data are boundary measurements, namely the displacement and the temperature. We study the stability of this problem restricting our selfs to identification of a straight emergent crack in a static thermoelastic medium. In our numerical approach to identify such cracks, we split the process into two steps; we identify the direction then the length of the crack. We study the uniqueness of the solution of the crack identification problem in a more general case; we prove identifiabilty of a crack modeled by a smooth curve via a non static thermoelastic model.

MOTS-CLÉS : problème inverse, thermoélasticité, stabilité, identifiabilité, dérivée de forme

KEYWORDS : inverse problem, thermoelasticity, stability, identifiability, shape derivative

1. Thermoelastic model

We consider a thermoelastic, isotropic and homogeneous material in a bounded smooth domain $\Omega \subset \mathbb{R}^2$. Σ is a crack included in $\overline{\Omega}$, it is an injective regular curve. We denote by

$$I = [0, t^*], \text{the time interval and} \quad Q_\Sigma = (\Omega \setminus \Sigma) \times I. \tag{1}$$

The thermoelastic mathematical model is as follows ([3]):

$$\begin{array}{l} \operatorname{div}(\sigma(u)) = \gamma \operatorname{grad} T \quad \text{in} \quad Q_\Sigma. \\ \rho_0 C \dfrac{\partial T}{\partial t} + \theta_0 \gamma \dfrac{\partial}{\partial t} \operatorname{div}(u) = k \Delta T \quad \text{in} \quad Q_\Sigma. \\ (\sigma(u).n - \gamma T n, \dfrac{\partial T}{\partial n}) = (F, \Phi) \quad \text{on} \quad \Gamma_N \times I. \\ (\sigma(u).n - \gamma T n, \dfrac{\partial T}{\partial n}) = (0, 0) \quad \text{on} \quad \Sigma \times I. \\ (u, T) = (0, 0) \quad \text{on} \quad \Gamma_D \times I. \\ u(x, 0) = u_0, \quad \dfrac{\partial u}{\partial t}(x, 0) = 0, \quad x \in \Omega \setminus \Sigma. \\ T(x, 0) = T_0, \quad \triangle T_0 = 0, \quad x \in \Omega \setminus \Sigma. \end{array} \tag{2}$$

Where Γ_D and Γ_N are a partition of $\partial\Omega$; F is a load and Φ is a heat flux applied on a part of the boundary. n is the exterior normal vector to $\partial(\Omega \setminus \Sigma)$

The unknowns of the direct problem defined by the thermoelastic model (2) are the displacement u and the temperature T while the unknown of our inverse problem is the geometry Σ.

2. Identifiability

Our inverse problem consists in identifying a crack Σ, modeled by a smooth curve, inside a thermoelastic material, using boundary measurements. Namely, we apply a load F and a heat flux Φ on a part of the boundary and we measure the displacement u^m and the temperature T^m on a part of the boundary $\Gamma_M \subset \Gamma_N$. It's a non-destructive control problem: The loading does not generate a propagation of the crack, we remain in the assumption of small deformations.

The identifiabilty is the proof of the uniqueness of the crack that corresponds to the data (u^m, T^m). We prove an identifiability result in the thermoelastic case following the main steps of standard identifiability results for unknown geometries (see [1],[4]) but the tools make a major difference.

Theorem 1. *Let Σ and $\tilde{\Sigma}$ be two cracks inside Ω and let (u, T) and $(\tilde{u}, \tilde{T})$ be the solutions of (2) respectively in Q_Σ and $Q_{\tilde{\Sigma}}$, with a load $F \neq 0$ ($F \in H^{-1/2}(\Gamma_N)^2$). Assuming that at least one of the stress intensity factors, K_I or K_{II} is not equal to zero (solution is not regular in a neighborhood of the crack tips), then if Σ and $\tilde{\Sigma}$ result into the same measurements on Γ_M , we have $\Sigma = \tilde{\Sigma}$.*

Remark 1. *The assumption "$K_I \neq 0$ or $K_{II} \neq 0$" is not restrictive; in fact this assumption expresses the physical existence of the crack. If bouth stress intensity factors vanish there is no separation or sliding of the crack lips. We do not impose any additional assumption on the regularity of the temperature in the vicinity of the crack either on the heat flux Φ.*

3. Stability

Stability is a crucial question when studying an inverse problem. Since measurements are vitiated by experimental errors, one have to make sure that small errors on measurements will not induce large errors on the solution of the inverse problem computed via these meassurements. To our knowledge stability results for inverse problems in mechanics are not frequent in literature. In this section we restrict our study of stability to the identification of an emergent straight crack via the static thermoelastic problem. We study the error that can be induced on the direction or on the length of the crack. Our study can be seen as an extension of results in [2], where authors prove directional stability estimates for cavity identification. Mathematically the stability is the continuity of the operator which corresponds the geometry to the measurements:

$$\eta^{-1}(\Sigma_{ad}) \to \Sigma_{ad}$$

The main tool in our stability proof is the *shape derivative* which become a classical tool in the study of geometric inverse problems.
The basic idea is to consider perturbations of the studied domain via a family of diffeomorphism: $F^h : \Omega \to \mathbb{R}^2$ defined by:

$$F^h = Id + h\Theta \tag{3}$$

where h is a positive constant (supposed to be small) and $\Theta \in W^{1,\infty}(\overline{\Omega})$ is the velocity field. We denote by $\Omega_h = F^h(\Omega)$. For u_h defined in Ω_h we associate u^h defined in Ω by:

$$u^h = u_h o F^h, \tag{4}$$

The general form of our stability results is:

Theorem 2. *We assume that at least one stress intensity factor K_I or K_{II} does not vanish and that $\Gamma_M \subset \Gamma_N$ has a strictly positive measure, then*

$$\lim_{h\to 0} \frac{\|u^h - u\|_{L^2(M)}}{h} > 0. \tag{5}$$

We develop two stability results one with respect to the direction and the other with respect to the length. For each result a convinient choice of Θ is made.

4. A numerical approach to identify straight emergent crack

We develop a numerical approach to identify a straight emergent crack in a thermoelastic medium satisfying the model (2) in its static case. The main tool in our process to identify the fracture is the reciprocity gap principle introduced by Andrieux and Ben Abda [6] for identification of plane cracks in the case of a 3D Laplace model. This tool was used by BenAbda, BenAmeur and Jaoua [1] for the identification of 2D straight crack in a linear elastic domain and later by Andrieux and Bui [5] to identify the plane containing the fracture via a 3D transient isotropic thermoelastic model.
Since we deal with a 2D problem, identifying such crack is equivalent to identify its direction and its length. We split our identification problem into two steps, in the first step

we identify the direction of the crack and in a second one we identify its length. We denote by $W = \{v \in H^1(\Omega); \text{div}\sigma(v) = 0 \quad in \quad \Omega \setminus \Sigma\}$. The reciprocity gap functional corresponding to the considered problem is defined on W by:

$$RG(v) = \int_{\partial\Omega} Fv - R(\varepsilon(v))Nu \tag{6}$$

4.1. Identification of the crack direction

$$RG(v) = \int_{\Sigma} R : \varepsilon(v)N.[\![u]\!] - \int_{\Omega\setminus\Sigma} \gamma T \text{div}(v)$$

where $N = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ is the unit normal vector to Σ.

Particular choices of fields v in W allow to reduce the expression of RG into an integral on the boundary of Ω and to define a system of equations satisfied by N:

$$\begin{cases} (N_1S_1 - N_2S_2) = \dfrac{1+\nu}{E} RG(v_1) \\ (N_2S_1 + N_1S_2) = \dfrac{1+\nu}{E} RG(v_2) \\ (N_1S_1 + N_2S_2)^2 = (\dfrac{1+\nu}{E})^2 (RG(v_1)^2 + RG(v_2)^2) \\ N_1^2 + N_2^2 = 1 \end{cases} \tag{7}$$

Solving (7) allows to identify N_1 and N_2 and so the crack direction.

4.2. Identification of the crack length

We denote by ℓ the exact length of the crack Σ that we want to identify. Let $\tilde{\Sigma}$ be a given candidate crack of length $\tilde{\ell}$, the singular part of the corresponding displacement field defined in a polar coodinates system of origin the interior crack tip is:

$$S_{\tilde{\ell}} = \frac{\sqrt{\rho}}{\sqrt{2\pi}E} \begin{pmatrix} cos(\dfrac{\varphi}{2})(3 - \nu - (1+\nu)cos\varphi) \\ sin(\dfrac{\varphi}{2})(3 - \nu - (1+\nu)cos\varphi) \end{pmatrix}$$

$S_{\tilde{\ell}}$ is in W and satisfies $\sigma(S_{\tilde{\ell}})n = 0$ on $\tilde{\Sigma}$.
We define a function rg by $rg(\tilde{\ell}) = RG(S_{\tilde{\ell}})$.

Lemma 1. *If ℓ is the length of the crack Σ to identify, then*

$$RG(S_\ell) = 0.$$

We develop an algoritm leading to identify the crack length as the zero of the reciprocity gap function RG

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