# Recovery of parameters of the coupled heart-torso model in cardiac electrophysiology ⋆

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**Abstract.** In this paper, using the art model in cardiac electrophysiology, the monodomain model, which describes the electrical activity in cardiac tissue, we prove a stability estimates of parameters identification problem. For this purpuse, we first establish a Carleman estimate for the coupled heart-torso system. By means of this estimation and following the Bukhgeim-klibaniv method we prove our main result which is a Lipschitz stability estimate of conductivities parameters.

**Keywords:** Lipschitz stability estimate · cardiac electrophysiology · Carleman estimate · conductivities parameters.

## 1 Introduction

We assume that the intra- and extracellular conductivities $\sigma_{\mathbf{i}}$ and $\sigma_{\mathbf{e}}$ are proportional. Let the bulk conductivity tensor of the medium and the transmembrane conductivity tensor defined respectively as follows

$$\sigma_{\mathbf{h}} = \sigma_{\mathbf{i}} + \sigma_{\mathbf{e}}, \tag{1.1}$$

and

$$\sigma_{\mathbf{m}} = \sigma_{\mathbf{i}} \sigma_{\mathbf{h}}^{-1} \sigma_{\mathbf{e}}. \tag{1.2}$$

We assume that the cardiac domain $\Omega_{\mathbf{h}}$ is an open bounded subset with locally Lipschitz continuous boundary of $\mathbb{R}^3$ and the torso domain is occupied by $\Omega_{\mathbf{t}}$. We denote by $S$ the interface between both domains $\Omega_{\mathbf{h}}$ and $\Omega_{\mathbf{t}}$, by $\Gamma_{\text{ext}}$ the external boundary of $\Omega_{\mathbf{t}}$ and by $n$ the outward unit normal to $\Omega_{\mathbf{t}}$. Let $S^+$ (resp. $S^-$ ) be the part of $S$ corresponding to the positive (resp. negative) direction of the normal $n$. We define the global domain $Q = \Omega \times (0, T)$ where $\Omega = \overline{\Omega_{\mathbf{h}}} \cup \Omega_{\mathbf{t}}$. ( see figure 1.)

The system of equations modeling the electrical activity in the heart is

$$\begin{cases} \partial_t v_{\mathbf{m}} - \operatorname{div}(\sigma_{\mathbf{m}} \nabla v_{\mathbf{m}}) = I_{app} + I_{ion}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}) & \text{in } Q_{\mathbf{h}} := \Omega_{\mathbf{h}} \times (0, T), \\ \operatorname{div}\big(\sigma_{\mathbf{h}} \nabla u_{\mathbf{h}}\big) = -\operatorname{div}(\sigma_{\mathrm{i}} \nabla v_{\mathbf{m}}) & \text{in } Q_{\mathbf{h}}, \\ \partial_t \mathbf{w} - \mathbf{F}(v_{\mathbf{m}}, \mathbf{w}) = 0 & \text{in } Q_{\mathbf{h}}, \\ \partial_t \mathbf{z} - \mathbf{G}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}) = 0 & \text{in } Q_{\mathbf{h}}. \end{cases} \tag{1.3}$$

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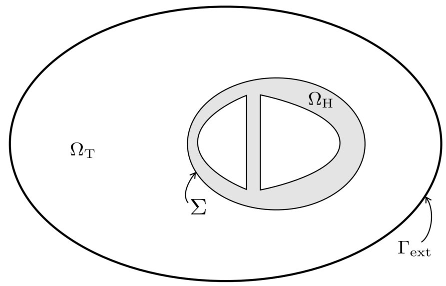


**Fig. 1.** The heart and torso domains.

Here, the transmembrane potential $v_{\mathbf{m}}$ is defined as follows

$$v_{\mathbf{m}} = u_{\mathbf{i}} - u_{\mathbf{h}}, \tag{1.4}$$

where $u_{\mathbf{i}}$ and $u_{\mathbf{h}}$ are the intra- and extra- cellular potentials. The $I_{app}$ is an external applied electrical current and $I_{ion}$ is the ionic current across the membrane which is defined as follows

$$I_{ion}(\overline{\varrho}, v, \mathbf{w}, \mathbf{z}) = \sum_{i=1}^{N} \overline{\varrho}_{\mathbf{i}} y_{\mathbf{i}}(v) \prod_{j=1}^{k} (\tilde{w}_l)_j^{p_{j,i}} (v - E_{\mathbf{i}}(\mathbf{z})), \tag{1.5}$$

where

$$E_{\mathbf{i}}(\mathbf{z}) = \overline{\gamma}_{\mathbf{i}} log\Big(\frac{z_{\mathbf{e}}}{z_{\mathbf{i}}}\Big), \quad \mathbf{z} = (z_1, \ldots, z_{\mathbf{m}}). \tag{1.6}$$

Here $\overline{\gamma}_{\mathbf{i}}$ is a constant and we denotes by $z_{\mathbf{i}}$, $i = 1, \ldots, m$ and $z_{\mathbf{e}}$ the intra- and extracellular concentration.

We define the evolution of the gating variables $\mathbf{w} := (w_1, \ldots, w_k)$ and the ionc intacellular concentrations $\mathbf{z} := (z_1, \ldots, z_{\mathbf{m}})$ by the following functions $F(v_{\mathbf{m}}, \mathbf{w})$ and $G(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z})$ which are defined as follows

$$\partial_t w_j = F_j(v_{\mathbf{m}}, w_j) := \alpha_j(v_{\mathbf{m}})(1 - w_j) - \beta_j(v_{\mathbf{m}}) w_j, \quad j = 1, \ldots, k, \tag{1.7}$$

where $\alpha_j$ and $\beta_j$ are a positive function with $0 \leq w_j \leq 1$ and

$$\partial_t z_{\mathbf{i}} = G_{\mathbf{i}}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}) := -J_{\mathbf{i}}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \log z_{\mathbf{i}}) + H_{\mathbf{i}}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}), \qquad \forall i = 1, \ldots, m, \tag{1.8}$$

where

$$\begin{aligned} & J_{\mathbf{i}} \in C^2(\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^k \times \mathbb{R}), \quad 0 < g_*(\mathbf{w}) \leq \frac{\partial J_{\mathbf{i}}}{\partial \tau}(\overline{\rho}, v_{\mathbf{m}}, \mathbf{w}, \tau) \leq g^*(\mathbf{w}), \\ & \Big| \frac{\partial J_{\mathbf{i}}}{\partial \tau}(\overline{\rho}, v_{\mathbf{m}}, \mathbf{w}, 0) \Big| \leq L_v(\mathbf{w}), \text{ with} g_*, g^*, L_v \text{ belong to} \, C^1(\mathbb{R}^k, \mathbb{R}_+) \text{ and} \end{aligned} \tag{1.9}$$

$$H_{\mathbf{i}} \in C^2(\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}^k \times (0,+\infty)^m) \cap Lip(\mathbb{R}_+^* \times \mathbb{R} \times [0,1]^k \times (0,+\infty)^m). \quad (1.10)$$

In order that the intra cellular current does not diffuse outside the heart, we should add the following condition on the interface boundary $\Sigma = S \times (0,T)$

$$\sigma_{\mathbf{m}} \nabla v_{\mathbf{m}} \cdot n = 0 \quad \text{on } \Sigma. \tag{1.11}$$

Our mathematical model is based on the coupling of (1.3) with the following diffusion equation in $Q_{\mathbf{t}} = \Omega_{\mathbf{t}} \times (0,T)$

$$\operatorname{div}(\sigma_{\mathbf{t}} \nabla u_{\mathbf{t}}) = 0 \quad \text{in } Q_{\mathbf{t}}, \tag{1.12}$$

with the following condition on the external boundary $\Sigma_{\text{ext}} = \Gamma_{\text{ext}} \times (0,T)$ which is assumed to be isolated

$$\sigma_{\mathbf{t}} \nabla u_{\mathbf{t}} \cdot n = 0 \quad \text{on } \Sigma_{\text{ext}}, \tag{1.13}$$

where $u_{\mathbf{t}}$ and $\sigma_{\mathbf{t}}$ represent the torso potential and the conductivity tensor of the torso. In order to diffuse informations potentials and currents from the heart to thorax, we need to introduce the following transmission conditions

$$\begin{cases} u_{\mathbf{h}} = u_{\mathbf{t}} & \text{on } \Sigma, \\ \sigma_{\mathbf{h}} \nabla u_{\mathbf{h}} \cdot n = \sigma_{\mathbf{t}} \nabla u_{\mathbf{t}} \cdot n & \text{on } \Sigma. \end{cases} \tag{1.14}$$

To sum up, from (1.3) - (1.11) - (1.12) - (1.13) and (1.14), we obtain the following the coupled heart-torso model

$$\begin{cases} \partial_t v_{\mathbf{m}} - \operatorname{div}(\sigma_{\mathbf{m}} \nabla v_{\mathbf{m}}) = I_{app} + I_{ion}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}) & \text{in } Q_{\mathbf{h}}, \\ \operatorname{div}\big(\sigma_{\mathbf{h}} \nabla u_{\mathbf{h}}\big) = -\operatorname{div}(\sigma_{\mathrm{i}} \nabla v_{\mathbf{m}}) & \text{in } Q_{\mathbf{h}}, \\ \operatorname{div}\big(\sigma_{\mathbf{t}} \nabla u_{\mathbf{t}}\big) = 0 & \text{in } Q_{\mathbf{t}}, \\ \partial_t \mathbf{w} - \mathbf{F}(v_{\mathbf{m}}, \mathbf{w}) = 0 & \text{in } Q_{\mathbf{h}}, \\ \partial_t \mathbf{z} - \mathbf{G}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}) = 0 & \text{in } Q_{\mathbf{h}}, \end{cases} \tag{1.15}$$

with the following interface conditions

$$\begin{cases} \sigma_{\mathbf{m}} \nabla v_{\mathbf{m}} \cdot n = 0 & \text{on } \Sigma, \\ u_{\mathbf{h}} = u_{\mathbf{t}} & \text{on } \Sigma, \\ \sigma_{\mathbf{h}} \nabla u_{\mathbf{h}} \cdot n = \sigma_{\mathbf{t}} \nabla u_{\mathbf{t}} \cdot n & \text{on } \Sigma, \end{cases} \tag{1.16}$$

and the following external boundary condition

$$\sigma_{\mathbf{t}} \nabla u_{\mathbf{t}} \cdot n = 0 \quad \Sigma_{\text{ext}}. \tag{1.17}$$

## 2 Global Carleman estimate for the coupled heart-torso system

In order to study our inverse problem, we should establish the global Carleman estimate for the coupled heart-torso model which is the key point. We consider now the following system :

$$\begin{cases} \partial_t v_{\mathbf{m}} - \operatorname{div}(\sigma_{\mathbf{m}} \nabla v_{\mathbf{m}}) = g & \text{in } Q_{\mathbf{h}}, \\ \operatorname{div}\big(\sigma_{\mathbf{h}} \nabla u_{\mathbf{h}}\big) = f_{\mathbf{h}} & \text{in } Q_{\mathbf{h}}, \\ \operatorname{div}\big(\sigma_{\mathbf{t}} \nabla u_{\mathbf{t}}\big) = f_{\mathbf{t}} & \text{in } Q_{\mathbf{t}}, \\ \partial_t \mathbf{w} - \mathbf{F}(v_{\mathbf{m}}, \mathbf{w}) = 0 & \text{in } Q_{\mathbf{h}}, \\ \partial_t \mathbf{z} - \mathbf{G}(\overline{\varrho}, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z}) = 0 & \text{in } Q_{\mathbf{h}}, \end{cases} \tag{2.1}$$

with the following interface conditions

$$\begin{cases} \sigma_{\mathbf{m}} \nabla v_{\mathbf{m}} \cdot n = 0 & \text{on } \Sigma, \\ u_{\mathbf{h}} = u_{\mathbf{t}} & \text{on } \Sigma, \\ \sigma_{\mathbf{h}} \nabla u_{\mathbf{h}} \cdot n = \sigma_{\mathbf{t}} \nabla u_{\mathbf{t}} \cdot n & \text{on } \Sigma, \end{cases} \tag{2.2}$$

and the following external boundary condition

$$\sigma_{\mathbf{t}} \nabla u_{\mathbf{t}} \cdot n = 0 \quad \Sigma_{\text{ext}}. \tag{2.3}$$

**Theorem 1.** *(Carleman estimate) There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ there exist $s_0 := s_0(\lambda) > 0$ and $C_\lambda > 0$ such that the solution $(v_{\mathbf{m}}, u) \in H^{1,2}(Q_{\mathbf{h}}) \times H^1(Q)$ to the system (1.15)-(1.16)-(1.17) satisfies*

$$\begin{aligned} &\int_{Q_{\mathbf{h}}} \Big( (s\varphi)(|\partial_t v_{\mathbf{m}}|^2 + |div(\sigma_{\mathbf{m}} \nabla v_{\mathbf{m}})|^2) + (s\varphi)^5 |v_{\mathbf{m}}|^2 + (s\varphi)^3 |\nabla v_{\mathbf{m}}|^2 \Big) e^{-2s\eta} dx dt \\ &\qquad + \int_Q \Big( (s\varphi)^3 |u|^2 + (s\varphi) |\nabla u|^2 \Big) e^{-2s\eta} dx dt \\ &\leq C_\lambda \Big( \int_{Q_{\mathbf{h}}} (s\varphi)^2 |g|^2 e^{-2s\eta} dx dt + \int_Q |f|^2 e^{-2s\eta} dx dt + \int_{\Sigma_{ext}} (s\varphi) |\nabla_\tau u_{\mathbf{t}}|^2 e^{-2s\eta} dx dt \\ &\qquad + \int_{\omega \times (0,T)} \big( (s\varphi)^5 |v_{\mathbf{m}}|^2 + (s\varphi)^3 |u_{\mathbf{h}}|^2 \big) e^{-2s\eta} dx dt \Big), \end{aligned} \tag{2.4}$$

*for any $s > s_0$, $g \in L^2(Q_{\mathbf{h}})$ and $f = (f_{\mathbf{h}} \chi_{\mathbf{h}} + f_{\mathbf{t}} \chi_{\mathbf{t}}) \in L^2(Q)$.*

## 3 Inverse problem for conductivity coefficients

Let $\omega \subset \Omega_{\mathbf{h}}$ be a non-empty subdomain of $\Omega_{\mathbf{h}}$, then there exists a weight function $\beta \in C^0(\overline{\Omega})$, $\beta_{\mathbf{i}} = \beta_{|\Omega_{\mathbf{i}}} \in C^2(\Omega_{\mathbf{i}})$ with $\mathbf{i} = \mathbf{h}$ or $\mathbf{t}$ satisfied some conditions. We denote $t \in (0, T)$ and $t_0 = T/2$. and we consider two sets of coefficients $(\sigma_{\mathbf{m}}, \sigma_{\mathbf{h}}, \sigma_{\mathbf{i}}, \sigma_{\mathbf{t}})$ and $(\widehat{\sigma}_{\mathbf{m}}, \widehat{\sigma}_{\mathbf{h}}, \widehat{\sigma}_{\mathbf{i}}, \widehat{\sigma}_{\mathbf{t}})$ and the corresponding solutions $(u, v_{\mathbf{m}}, \mathbf{w}, \mathbf{z})$ and $(\widehat{u}, \widehat{v}_{\mathbf{m}}, \widehat{\mathbf{w}}, \widehat{\mathbf{z}})$ of (1.15)-(1.16)-(1.17). Let $\alpha$ be a given smooth positive function $\alpha(x) \geq \alpha_0$, $x \in \overline{\Omega}_{\mathbf{h}}$ we define the the following sets of admissible coefficients :

$$\mathcal{A}_\alpha^{\mathbf{h}} = \{(\sigma_{\mathbf{i}}, \sigma_{\mathbf{e}}) \in C^2(\overline{\Omega}_{\mathbf{h}}),\ \sigma_{\mathbf{i}} \geq c_{\mathbf{i}} > 0,\ \sigma_{\mathbf{e}} \geq c_{\mathbf{e}} > 0 \text{ and } \sigma_{\mathbf{i}} = \alpha \sigma_{\mathbf{e}}\}, \tag{3.1}$$

$$\mathcal{A}_\alpha^{\mathbf{t}} = \{\sigma_{\mathbf{t}} \in C^2(\overline{\Omega}_{\mathbf{t}}),\ \sigma_{\mathbf{t}} \geq c_{\mathbf{t}} > 0\}, \tag{3.2}$$

for some positive constants $c_{\mathbf{i}}$, $c_{\mathbf{e}}$ and $c_{\mathbf{t}}$. In order to formulate our stability and uniqueness results of conductivities, we need to introduce the following assumptions :

***Assumption (A.1)*** There exists a constants $c_0 > 0$ such that

$$0 < c_0 < |\nabla\beta(x) \cdot \nabla\widehat{d}(x,t_0)|, \quad \text{in} \quad \overline{\Omega}\backslash\omega_0, \quad \omega_0 \subset \omega, \tag{3.3}$$

with $\widehat{d} \in \{\widehat{v}_{\mathbf{m}}, \widehat{u}_{\mathbf{h}}, \widehat{u}_{\mathbf{t}}\}$.

***Assumption (A.2)*** There exists a constants $M > 0$ such that

$$\|\widehat{v}_{\mathbf{m}}\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega_{\mathbf{h}}))} + \|\widehat{u}_{\mathbf{h}}\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega_{\mathbf{h}}))} + \|\widehat{u}_{\mathbf{t}}\|_{W^{2,\infty}(0,T;W^{2,\infty}(\Omega_{\mathbf{t}}))} \leq M. \tag{3.4}$$

**Theorem 2.** *(Stability) We assume that (**A.1**) and (**A.2**) are satisfied. Then, there exists a positive constant $C > 0$ depending on $\Omega, T, t_0, M$, such that*

$$\begin{aligned} &\|\sigma_{\mathbf{i}} - \widehat{\sigma}_{\mathbf{i}}\|_{H^1(\Omega_{\mathbf{h}})} + \|\sigma_{\mathbf{e}} - \widehat{\sigma}_{\mathbf{e}}\|_{H^1(\Omega_{\mathbf{h}})} + \|\sigma_{\mathbf{t}} - \widehat{\sigma}_{\mathbf{t}}\|_{H^1(\Omega_{\mathbf{t}})} \\ &\qquad \leq C\Big(\|(v_{\mathbf{m}} - \widehat{v}_{\mathbf{m}})(\cdot,t_0)\|_{H^2(\Omega_{\mathbf{h}})} + \|(u_{\mathbf{t}} - \widehat{u}_{\mathbf{t}})(\cdot,t_0)\|_{H^2(\Omega_{\mathbf{t}})} \\ &\qquad + \|(\boldsymbol{w} - \widehat{\boldsymbol{w}})(\cdot,t_0)\|_{H^1(\Omega_{\mathbf{h}})} + \|(\boldsymbol{z} - \widehat{\boldsymbol{z}})(\cdot,t_0)\|_{H^1(\Omega_{\mathbf{h}})} + \|\sigma_{\mathbf{i}} - \widehat{\sigma}_{\mathbf{i}}\|_{H^2(\omega)} \\ &+ \|v_{\mathbf{m}} - \widehat{v}_{\mathbf{m}}\|_{H^2(0,T;L^2(\omega))} + \|u_{\mathbf{h}} - \widehat{u}_{\mathbf{h}}\|_{H^2(0,T;L^2(\omega))} + \|u_{\mathbf{t}} - \widehat{u}_{\mathbf{t}}\|_{H^2(0,T;H^1(\Gamma_{ext}))}\Big), \end{aligned} \tag{3.5}$$

*for any $(\sigma_{\mathbf{i}}, \sigma_{\mathbf{e}}), (\widehat{\sigma}_{\mathbf{i}}, \widehat{\sigma}_{\mathbf{e}}) \in \mathcal{A}_\alpha^{\mathbf{h}}$, $\sigma_{\mathbf{t}}, \widehat{\sigma}_{\mathbf{t}} \in \mathcal{A}_\alpha^{\mathbf{t}}$ satisfying $(\partial^\alpha\sigma_{\mathbf{i}}, \partial^\alpha\sigma_{\mathbf{e}}) = (\partial^\alpha\widehat{\sigma}_{\mathbf{i}}, \partial^\alpha\widehat{\sigma}_{\mathbf{e}})$ on $S$, $|\alpha| \leq 1$ and $\sigma_{\mathbf{t}} > \sigma_{\mathbf{h}}$ on $S$.*

As a consequence, we can drive the following uniqueness result

**Corollary 1.** *(Uniqueness) Let us consider the same assumptions in Theorem 2 and let $(v_{\mathbf{m}}, u_{\mathbf{t}}, \boldsymbol{w}, \boldsymbol{z}) = (\widehat{v}_{\mathbf{m}}, \widehat{u}_{\mathbf{t}}, \widehat{\boldsymbol{w}}, \widehat{\boldsymbol{z}})$ at a fixed time $t_0$, $(v_{\mathbf{m}}, u_{\mathbf{h}}) = (\widehat{v}_{\mathbf{m}}, \widehat{u}_{\mathbf{h}})$ in $\omega \times (0,T)$, $u_{\mathbf{t}} = \widehat{u}_{\mathbf{t}}$ in the external boundary $\Sigma_{ext}$ and $\sigma_{\mathbf{i}} = \widehat{\sigma}_{\mathbf{i}}$ in $\omega_0$. Then, we have the following uniqueness result*

$$(\sigma_{\mathbf{i}}, \sigma_{\mathbf{e}}) = (\widehat{\sigma}_{\mathbf{i}}, \widehat{\sigma}_{\mathbf{e}}) \quad \textit{in} \quad \Omega_{\mathbf{h}}, \quad \textit{and} \quad \sigma_{\mathbf{t}} = \widehat{\sigma}_{\mathbf{t}} \quad \textit{in} \quad \Omega_{\mathbf{t}}. \tag{3.6}$$

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