Théorie des Volumes Finis Mixtes Hybrides et Application aux Aquifères

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RÉSUMÉ. Nous présentons une variante des éléments finis mixtes hybrides bâtie sur l'idée de base des volumes finis. Nous montrons que comme dans le cas des éléments finis mixtes hybrides la matrice associée au problème discret est symétrique et définie positive. En utilisant une version discrète de l'inégalité de Poincaré-Friedrichs nous montrons la stabilité et établissons des estimations d'erreurs pour les potentiels au centre des mailles, les potentiels et les flux sur les interfaces. Des résultats numériques sont présentés pour un aquifère fictif et un aquifère réel.

ABSTRACT. We present a variant of mixed hybrid finite element method using basic ideas of finite volume method. We show that as in mixed hybrid finite element method, solving the associated discrete system reduces to the solution of a symmetric positve definite linear system where interface discrete potentials are unknowns. Using a discrete Poincaré-Friedrichs type inequality we show stability and derive error estimates for interface potential, cell potential and fluxes. Numerical results for fictitious and real-life aquifers are presented.

MOTS-CLÉS : Volumes finis mixtes hybrides, écoulement, aquifère, simulation, stabilité, estimations d'erreurs.

KEYWORDS : mixed hybrid finite volume, flow, aquifers, simulation, stability, error estimates.

Introduction

Our intent in this paper is to propose a based mixed hybrid finite volume formulation for diffusion problems in general and single-phase flow in geological reservoir in particular. An important feature of our numerical approach is that it can handle external effects such as the presence of a lake on the interface of two adjacent grid-blocks or a river moving simultaneously on grid-blocks and along grid-blocks interfaces.

An accurate computation of the flow velocity is commonly needed for a good understanding of the transportation of contaminants in aquifers via simulation process. In this respect our numerical approach involves a formulation in two fields, namely pressure and velocity, and satisfies the flux continuity condition that is necessary to get a locally conservative scheme.

Our paper is organized as follows : the first section deals with the mathematical model. The second section presents the discretization process, stability and error bound estimates. The third section develops an application for flow in aquifers with hydraulic connection to lakes or rivers. Numerical simulations of fictitious and real-life aquifers, with comparison to exact and experimental solutions, are performed. The last section is devoted to the conclusion.

1. Model problem

Let $\Omega \subset \mathbb{R}^2$ be a polygon whose boundaries are parallel to orthogonal coordinates axes. Given f and T respectively a real-valued function and a matrix-valued function defined on Ω, one seeks an approximation of functions u(pressure) and q(velocity) such that :

$$q(x) = -T(x) grad\, u, \quad div\, q(x) = f(x) \quad \text{in} \quad \Omega \tag{1}$$

$$u = 0 \quad \text{on} \quad \Gamma_{Dir}, \quad q.n = 0 \ \text{on} \ \Gamma_{Neu} \tag{2}$$

where n is the unit normal vector of Γ outward to Ω. In the sequel we assume (except where otherwise is stated) that there exists T^-, T^+ real numbers such that :
$0 < T^- \leq T(x) \leq T^+$ a.e. in Ω, $f \in L^2(\Omega)$, $mes(\Gamma_{Dir}) > 0$, where mes(.) denotes Lebesgue measure in 1D. The additional assumption for $T(x)$, but very realistic for several applications, is that : $T(x)$ is piece-wise constant. Under these assumptions on data,one can prove that the system (1) - (2) possesses a unique variational solution.

2. Mixed Hybrid Finite Volume Analysis

2.1. Construction of the discrete system

Let's define a rectangular mesh on Ω denoted by $\{K\}_{K\in\mathbb{F}}$, where K is the generic name of mesh elements. Let $\mathbb{A}$ be the set of all edges of the previous mesh, $\mathbb{A}^{int}$ the subset of $\mathbb{A}$ made of interior edges, $\mathbb{A}^{Neu}$ the subset of $\mathbb{A}$ made of Neumann edges, $\mathbb{A}^{Dir}$ the subset of $\mathbb{A}$ made of Dirichlet edges, and $\mathbb{A}^{K}$ the subset of $\mathbb{A}$ made of K-edges. Denoting h_K the diameter of the mesh element K, we set : $h \equiv \max\limits_{K\in\mathbb{F}} h_K$. Supposing that the discontinuities of T coincide with mesh interfaces, let T_K be the uniform value of the diffusion coefficient over the mesh element K, Γ_K the boundary of K. Following [3] the mixed hybrid finite volume formulation of the system (1)-(2) consists in writing these governing equations under the following equivalent form :

$$div\, q = f,\ \ q(x) \ = \ -T_K \ \ grad\, u \ \ in \ \ K, \forall K \in \mathbb{F} \tag{3}$$

$$[|u|] = 0, \quad q.n_{K/L} + q.n_{L/K} \ = \ 0 \quad on \ \ \Gamma_K \cap \Gamma_L \ \ \forall\, K, L \in \mathbb{F} \text{ adjacent} \tag{4}$$

$$u = 0 \quad on \quad \Gamma_{Dir}, \quad q.n \ = \ 0 \quad on \quad \Gamma_{Neu} \tag{5}$$

where the symbol $[|u|]$ denotes the jump of u across any mesh interface, and $n_{K/L}$ the unit normal vector of $\Gamma_K \cap \Gamma_L$ outward to K. Integrating the first equation of (3) in K yields

$$\sum_{a\in\mathbb{A}^K} q_{a,K} \ = \ aire(K) < f >_K \qquad \forall K \in \mathbb{F} \tag{6}$$

where $< f >_K$ is the mean of f over K, $q_{a,K} = \int_a q \cdot n_{a,K} ds\ \forall a \in \mathbb{A}^K$ and where $n_{a,K}$ is the unit normal vector of the edge $a \in \mathbb{A}^K$ outward to K. Applying a numerical integration rule and taking into account the second equation of (3) yields for each $K \in \mathbb{F}$:

$$q_{a,K} \approx \frac{mes(a)T_K}{d(x_a, x_K)}[u_K - u_{a,K}] \qquad\qquad \forall a \in \mathbb{A}^K \tag{7}$$

where $u_K \equiv$ value of u at the center of K and $u_{a,K} \equiv$ value of u at the middle of $a \in \mathbb{A}^K$. From the continuity of u and flux on grid-block interfaces one has

$$u_{a,K} = u_{a,L}, \quad q_{a,K} + q_{a,L} = 0 \forall a \in \mathbb{A}^K \cap \mathbb{A}^L \text{ with } K \text{ and } L \text{ adjacent} \ . \tag{8}$$

On the other hand, one gets from (5) that, for each mesh element K adjacent to the domain boundary Γ,

$$u_{a,K} = 0 \quad \forall\, a \in \mathbb{A}^K \cap \mathbb{A}^{Dir}, \quad q_{a,K} = 0 \quad \forall\, a \in \mathbb{A}^K \cap \mathbb{A}^{Neu} \tag{9}$$

The equations (6)-(9) suggest the introduction of the following discrete problem : Find $\{U_K\}_{K\in\mathbb{F}}$, $\{U_{a,K}\}_{\substack{K\in\mathbb{F}\\ a\in\mathbb{A}^K}}$, $\{Q_{a,K}\}_{\substack{K\in\mathbb{F}\\ a\in\mathbb{A}^K}}$ such that :

$$\sum_{a\in\mathbb{A}^K} Q_{a,K} = aire(K) < f >_K \ \forall K \in \mathbb{F}, \tag{10}$$

$$Q_{a,K} = \frac{mes(a)T_K}{d(x_a, x_K)}[U_K - U_{a,K}] \forall a \in \mathbb{A}^K \forall K \in \mathbb{F}, \tag{11}$$

$$U_{a,K} = U_{a,L} \ \forall a \in \mathbb{A}^K \cap \mathbb{A}^L \text{with K and L adjacent}, \tag{12}$$

$$Q_{a,K} + Q_{a,L} = 0 \ \forall a \in \mathbb{A}^K \cap \mathbb{A}^L \quad \text{with K and L adjacent}, \tag{13}$$

$$U_{a,K} = 0 \quad \forall a \in \mathbb{A}^K \cap \mathbb{A}^{Dir} (\text{ Dirichlet conditions}), \tag{14}$$

$$Q_{a,K} = 0 \quad \forall a \in \mathbb{A}^K \cap \mathbb{A}^{Neu} (\text{ Neumann conditions}), \tag{15}$$

where $x_a \equiv$ middle point of the edge $a \in \mathbb{A}^K$, $x_K \equiv$ the centre of the mesh element K, $d(x_a, x_K) \equiv$ distance from x_a to x_K and, $aire(K) \equiv$ Lebesgue measure of K in 2D. Equations (10)-(15) define what we call hereafter a mixed hybrid finite volume scheme. (Note that eliminating the unknowns $\{U_{a,K}\}_{\substack{K\in\mathbb{F}\\ a\in\mathbb{A}^K}}$ from the mixed hybrid finite volume scheme (10)-(15) yields the classical cell-centered finite volume method as presented in [2]. A connection with mixed hybrid finite element as presented in [1] can be shown via numerical integration (see [4]). from (10) and (11) one gets

$$U_K = \frac{\sum_{a\in\mathbb{A}^K} \tau_{a,K} U_{a,K}}{\sum_{a\in\mathbb{A}^K} \tau_{a,K}} + \frac{aire(K)}{T_K \sum_{a\in\mathbb{A}^K} \tau_{a,K}} < f >_K \tag{16}$$

where $\tau_{a,K} = \dfrac{mes(a)}{d(x_a, x_K)}$. Setting $U_a = U_{a,K} \quad \forall a \in \mathbb{A}^K \quad \forall K \in \mathbb{F}$ and $\mathbb{A}^{int/Neu} = \mathbb{A}^{int} \cup \mathbb{A}^{Neu}$, one deduces from (12) and (13) that the $\{U_a\}_{a\in\mathbb{A}^{int/Neu}}$ verify the following linear system of equations :

$$\sum_{K\in\mathbb{F}/a\in\mathbb{A}^K} \ \sum_{b\in\mathbb{A}^K} \frac{T_K \tau_{a,K} \tau_{b,K}}{\sigma_K}[U_a - U_b] = 0 \qquad \forall a \in \mathbb{A}^{int/Neu} \tag{17}$$

with

$$U_a = 0 \qquad \forall a \in \mathbb{A}^{Dir} \tag{18}$$

where one has set $\sigma_K = \sum_{b\in\mathbb{A}^K} \tau_{b,K} \qquad \forall K \in \mathbb{F}$

Proposition 2.1 The discrete problem consisting to find $\{U_a\}_{a\in\mathbb{A}^{int/Neu}}$ satisfying (17)-(18) possesses a unique solution.

Proof Since the system (17)-(18) is a square a system, we need only prove that the following discrete problem : Find $\{U_a\}_{a\in\mathbb{A}^{int/Neu}}$ such that

$$\sum_{K\in\mathbb{F}/a\in\mathbb{A}^K}\ \sum_{b\in\mathbb{A}^K}\frac{T_K\tau_{a,K}\tau_{b,K}}{\sigma_K}[U_a-U_b]=0 \qquad \forall a\in\mathbb{A}^{int/Neu} \tag{19}$$

$$U_a=0 \qquad \forall a\in\mathbb{A}^{Dir} \tag{20}$$

admits zero-vector as a unique solution. Let $a\in\mathbb{A}^{int/Neu}$. Multiplying (19) by U_a, summing over $a\in\mathbb{A}^{int/Neu}$ and re-ordering the terms yields :

$$\sum_{K\in\mathbb{F}}\sum_{a,b\in\mathbb{A}^K}\frac{T_K\tau_{a,K}\tau_{b,K}}{\sigma_K}[U_a-U_b]^2=0.$$

Since $\text{mes}(\Gamma_{Dir})>0$, and thank to the connectivity of Ω one gets the result.

2.2. Stability and error estimates

We suppose here that $mes(\Gamma_{Neu})=0$, which implies that $\mathbb{A}^{Neu}=\emptyset$, $\mathbb{A}^{int/Neu}=\mathbb{A}^{int}$. We also suppose that all the edges $a\in\mathbb{A}$ are numbered and we denote $E(\mathbb{A})$ the space made of functions V taking a constant-value V_a on each interior edge a and zero on each boundary-edge. We identify each function V of $E(\mathbb{A})$ with a vector $(V_a)_{a\in\mathbb{A}}$, where V_a denotes the value of V on the edge a. From then on, to each vector $(V_a)_{a\in\mathbb{A}}$ of $E(\mathbb{A})$ is associated a vector-valued function $\vec{V}$ defined (almost everywhere) from Ω to $\mathbb{R}^4$ by : $\vec{V}(x)=(V_a)_{a\in\mathbb{A}^K}$ if $x\in K$, with $K\in\mathbb{F}$ Let's $E(\mathbb{A},\mathbb{F})$ be the space made of functions $\vec{V}$ and equipped with the discrete norm defined by

$$\left\|\vec{V}\right\|^2_{E(\mathbb{A},\mathbb{F})}=\sum_{K\in\mathbb{F}}\sum_{a,b\in\mathbb{A}^K}\frac{\tau_{a,K}\tau_{b,K}}{\sigma_K}[V_a-V_b]^2 \qquad \forall\vec{V}\in E(\mathbb{A},\mathbb{F}) \tag{21}$$

2.2.1. Stability

We will prove the stability of the scheme (17)-(18) in the sense of the norm (21). Assume there exists $0<\omega\le 1$ not depending on h s.t. $\omega h\le h_K\le h \quad \forall K$.

Lemma 2.1(Discrete Poincaré type inequality)

For each function $\vec{V}\in E(\mathbb{A},\mathbb{F})$ defined as stated above one has :

$$\sum_{K\in\mathbb{F}}(aire(K)\sum_{a\in\mathbb{A}^K}|V_a|^2)\le\frac{8\,[diam\,(\Omega)]^2}{\omega^4}\sum_{K\in\mathbb{F}}\sum_{a,b\in\mathbb{A}^K}\frac{\tau_{a,K}\tau_{b,K}}{\sigma_K}[V_a-V_b]^2$$

Proposition 2.2 (Stability result) The unique solution $\{U_a\}_{a\in\mathbb{A}^{int}}$ of the discrete problem verifies :

$$\sum_{K\in\mathbb{F},\, a,b\in\mathbb{A}^K} \frac{\tau_{a_K}\tau_{b_K}}{\sigma_K}\,[U_a - U_b]^2 \le \frac{8\,[diam\,(\Omega)]^2}{(T^-)^2\omega^2}\,\|f\|^2_{L^2(\Omega)} \qquad (22)$$

Proof of proposition 2.2 One deduces from (17) that

$$\sum_{K\in\mathbb{F},\, a,b\in\mathbb{A}^K} \frac{\tau_{a,K}\tau_{b,K}}{\sigma_K}\,[U_a - U_b]^2 = \sum_{a\in\mathbb{A}^{int}} \left\{ \sum_{K\in\mathbb{F},\, a\in\mathbb{A}^K} \int_K \left[f(x) U_a \frac{\tau_{a,K}}{\sigma_K} \right] dx \right\}$$

We obtain from a double application of Cauchy-Schwarz inequality that :

$$\sum_{K\in\mathbb{F},\, a,b\in\mathbb{A}^K} \frac{\tau_{a_K}\tau_{b_K}}{\sigma_K}\,[U_a - U_b]^2 \le \frac{\|f\|_{L^2(\Omega)}}{T^-\omega^2} \left\{ \sum_{K\in\mathbb{F}} aire(K) \left[\sum_{a\in\mathbb{A}^K} U_a^2 \right] \right\}^{1/2}$$

The inequality (22) follows by application of lemma 2.1.

2.2.2. Error Estimates in adequate discrete norms

In what follows we assume that $u_{|_K} \in C^2(\bar{K})$ for all K in $\mathbb{F}$.

• **Error estimate for interface potential** $\{u_{a,K}\}_{K\in\mathbb{F}, a\in\mathbb{A}^K}$

One assume that the mesh is made of square elements of size h, using the conservativity of the approximation of fluxes and a Young type inequality we can show that, setting $e_a = u_a - U_a \quad \forall a \in \mathbb{A}^{Dir}$,

$$\sum_{K\in\mathbb{F}} \left(\sum_{a,b\in\mathbb{A}^K} |e_a - e_b|^2 \right) \le Ch^2 \qquad (23)$$

•**Error estimate for cell center potential** $\{u_K\}_{K\in\mathbb{F}}$

By introducing ficticious mesh elements around the boundary of Ω one can show that, setting $e_K = u_K - U_K \quad \forall K \in \mathbb{F}$,

$$\sum_{K,L\in\mathbb{F}^*;\; K\cap L\in\mathbb{A}} |e_K - e_L|^2 \le Ch^2 \qquad (24)$$

• **Error estimate for interface flux** $\{q_{a,K}\}_{K\in\mathbb{F}, a\in\mathbb{A}^K}$

Applying the Young inequality yields

$$\sum_{K\in\mathbb{F}} \left(\sum_{a\in\mathbb{A}^K} |q_{a,K} - Q_{a,K}|^2 \right) \le Ch^2 \qquad (25)$$

Recall that C represents in the previous equations miscellaneous constants in $\mathbb{R}^*_+$, not depending on h.

3. Application to aquifers simulation

3.1. Simulation with fictitious aquifers

The main purpose here is to verify that the order of the numerical convergence confirms the one given by the theoretical results (23), (24) and (25).
We consider a 2D flow in the unit square $]0,1[\times]0,1[$ governed by the following system :

$$div\, q \;=\; 2\pi^2\, cos(\pi x_1)cos(\pi x_2) \qquad in \quad]0,1[\times]0,1[$$

$$q \;=\; -\; grad\, u \qquad in \quad]0,1[\times]0,1[$$

$$q.n \;=\; 0 \qquad on \qquad \Gamma_{north} \cup \Gamma_{east} \cup \Gamma_{west}$$

$$u = cos(\pi x_1) \qquad on \qquad \Gamma_{south}$$

The exact potential is given by : $u(x_1, x_2) \;=\; cos(\pi x_1)cos(\pi x_2)$
In figure 1 we compare the exact flow with the flow computed by mixed hybrid finite volume method at different level of refinement. In figure 2 we show the convergence rate in discrete H^1_0 norm for the cell center potential and grid-block interface potential. On the other hand we show the convergence rate in L^2 norm for the flux •

3.2. Simulation of Massoumbou aquifer

Massoumbou aquifer is the main drinking water supply for the city of Douala, the largest and the main industrial and commercial city in Cameroon. It is made up of 2 zones, the upper and the lower Massoumbou, almost horizontally seperated by a slightly permeable layer ; our numerical scheme was used to simulate the upper Massoumbou aquifer, which has 5 producing wells. For the simulation the following data obtained from BRGM[1] were used : on the left side of the river Dibamba the boundary hydraulic potential was set to 5.00 (Dirichlet boundary condition), while on the right side the no flow condition is used (Neumann boundary condition), the mean values of infiltrated flow rate from the rivers Dibamba and Wouri were set respectively to $5.0E-6m^3/s$ and $8.3E-\;6m^3/s$ per unit area ; the mean value of flow rate for producing wells was set to $0.07m^3/s$ per unit area ; the spacial distribution of transmissivities is given by figure 3 ; finally numerical simulation was carried out through a square grid (of size $h\;=\;400m$)

1. BRGM means Bureau Régional Géologique et Minier. We have consulted the BRGM report of 1981 titled "Alimentation de la ville de Douala en eaux souterraines"

involving a discretization of rivers. The discrete version of rivers lead to sources located either in grid-blocks or on grid-block intefaces (see figure 3)

Numerical results : We've taken the sea mean level as a reference level for hydraulic potential called again piezometric head. Figure 3 show the results computed with mixed hybrid finite volume method (respectively for the piezometric head contour and for the velocity field) when solving numerically the flow problem in upper Massoumbou aquifer. The comparison of our numerical results with experimentally measured hydraulic potential shows a very slight difference far from the producing wells (about 0.05m in absolute error and 2.5% in relative error). This difference becomes relatively significant when we compare with experimentally measured well hydraulic potential (about 3.62m in absolute error and 10% in relative error). From a practical point of view these results are satisfactory in the sense that for creating producing wells in an aquifer it is natural to operate where the piezometric head is maximal (taking the sea mean level as a reference level) and the flux is not negligible. So a high level of accuracy is required for the computed piezometric head and flux.

4. Conclusion

A mixed hybrid finite volume numerical method, for solving flow problems has been presented. Its formulation of a flow problem takes into account the main physical aspects of the flow as the local mass conservation, the continuity of the flux across grid-blocks interface. This method permits a simultaneous computation of the potential and the velocity thanks to the fact that it involves a two-field approach. The computed velocity is of the same order of precision as the computed pressure. This is very attractive when solving a fluid flow problem coupled with a transport equation. Error estimates have been given for interface potential and cell center potential in discrete H_0^1, and for flux across grid-blocks interface in L^2 norm. This theoretical result has been confirmed by numerical test on fictitious aquifers.

An application of our numerical approach has been carried out for solving a flow problem within a real-life aquifer, namely upper Massoumbou aquifer. The relative difference between the computed hydraulic potential and the experimentally measured one was almost 2.50% far from producing wells. This quantity was almost 10% in grid-blocks containing the producing wells. These results were considered as very satisfactory by SNEC engineers.

We intend to apply mixed hybrid finite volume method to solving transient flow within real-life aquifers with unstructured meshes •

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A. Figures

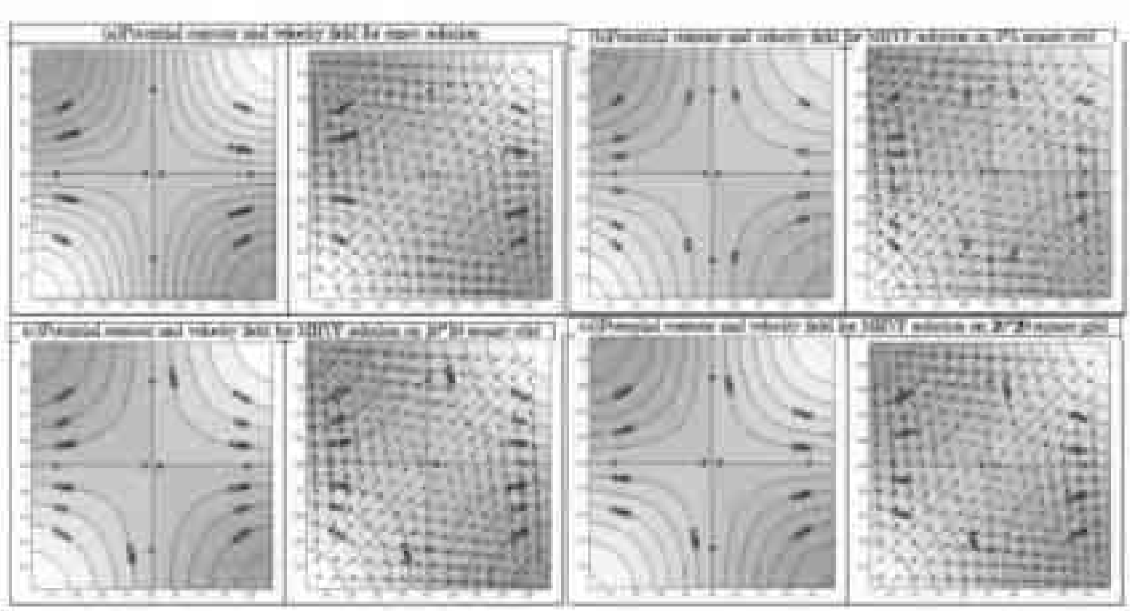

Figure 1. *Comparison of the exact solution with the MHFV solution for different levels of refinement*

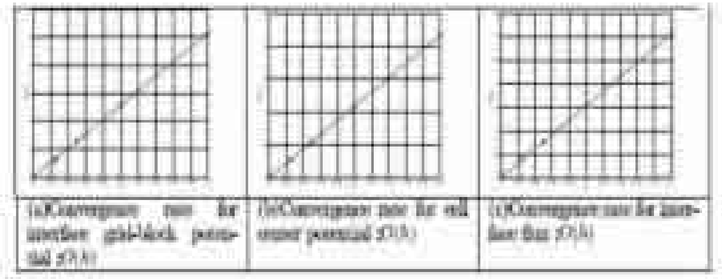

Figure 2. *Convergence rate*

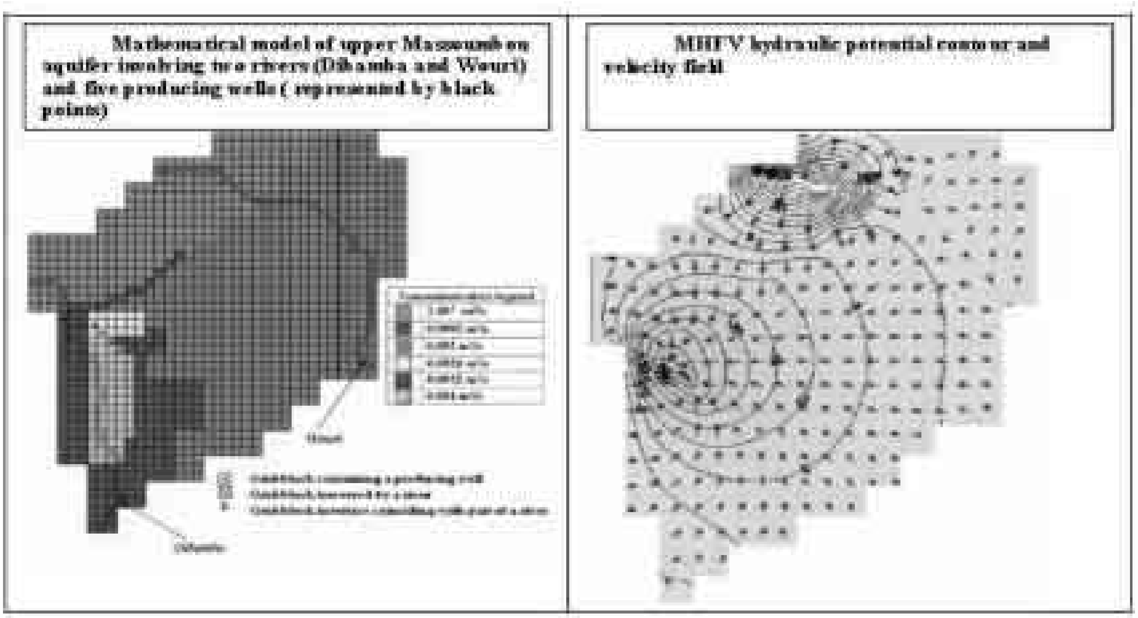

Figure 3. *Results on upper Massoumbou aquifer*