Finding polynomials to count lattice points; Computer explorations with MuPAD-Combinat.

Janvier Nzeutchap

Institut Gaspard Monge - Laboratoire d'Informatique, UMR CNRS 8049
Université de Marne-la-Vallée
77454 Marne-la-Vallée Cedex 2
FRANCE
email: jnzeutch@etudiant.univ-mlv.fr
http: monge.univ-mlv.fr/~nzeutcha

ABSTRACT. We are interested in Algebraic Combinatorics, a subject we give an overview, and in Symbolic Computation. In this paper we describe a problem we recently encountered in some experiments we are still carrying on. This problem is about generating polynomials counting lattice points in certain particular types of convex polytopes. The lattice points numbers are interpreted as Kostka numbers and Littlewood-richardson coefficients and they are of great interest in algebraic combinatorics. For some demonstrations, we make use of MuPAD-Combinat, an open-source algebraic combinatorics package for the computer algebra system MuPAD.

RÉSUMÉ. Nos travaux se situent dans les domaines de la Combinatoire Algébrique dont nous faisons une brève présentation, et du Calcul Symbolique ou Formel. Dans ce papier, nous présentons un problème posé à la suite de certains calculs que nous avons effectués tout récemment; calculs relatifs à la détermination des polynômes générateurs de nombres de Kostka et de coefficients de Littlewood-Richardson. Le problème en question porte plus généralement sur le degré réel d'un polynôme énumérant les points entiers dans un polytope convexe dilaté par un facteur entier. Nous effectuons quelques démonstrations de calculs à l'aide de MuPAD-Combinat, un package open source pour la combinatoire algébrique en MuPAD.

KEYWORDS : algebraic combinatorics, convex polytope, distributed computation, Kostka numbers, Littlewood-richardson coefficients, MuPAD-Combinat, stationary set of polynomials.

MOTS-CLÉS : calcul distribué, calcul formel, coefficients de Littlewood-Richardson, combinatoire algébrique, MuPAD-Combinat, nombres de Kostka, polytope convexe, suite stationnaire de polynômes.

Contents

1. Introduction

1.1. An overview on Algebraic Combinatorics

Situated halfway between mathematics and computer science, combinatorics, with some simplification, can be said to be the mathematics of the discrete and of the finite. Algebraic combinatorics consists in using techniques from algebra, topology and geometry, in the solution of combinatorial problems, and conversely, in using combinatorial methods to approach problems in those areas [4]. Although combinatorial mathematics has been pursed since time immemorial, and at a reasonable scientific level at least since Leonhard Euler (1707-1783), the subject has come into its own only in the last few decades [1]. One of the most basic properties of a finite collection of objects is its number of elements. Given an infinite sequence of sets $\{A_n\}_{n=0}^{+\infty}$ parameterized by n, sets of objects satisfying a set of combinatorial specifications, one is interested in how to compute the cardinality $a_n = |A_n|$ [4]. More precisely is there an efficient algorithm to generate or count those objects, is there any mathematical formulas? Algebraic combinatorics approaches such problems using generating functions and bijective constructions. Algebraic combinatorics very often deals with counting partitions of various kinds, meaning the number of ways to break an object into smaller objects of the same kind, the study of partitions was begun by Euler and is very active to this day [1].

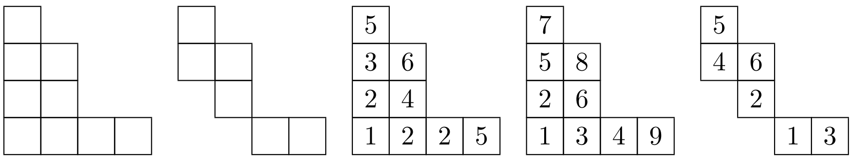
There are also strong and interesting connections between combinatorics and algebraic structures in general, such as monoid structures, symmetric functions, the study of irreducible representations of the symmetric group, etc. All those subjects are unfortunately too sophisticated to go into details here. Combinatorics is also used in many ways in computer science, for instance for the construction and analysis of various algorithms. Most of the work carried out in algebraic combinatorics is done using words on ordered alphabets. One can read [2] for an introduction to algebraic combinatorics on words.

1.2. Preliminary definitions

The following definitions are useful to understand the origin of the problem we are interested in, but not necessarily the more general problem itself.

Partitions, Ferrers diagrams and Young tableaux

A *partition* of a positive integer n is a way of writing n as a sum of weakly decreasing integers. For example $\lambda = (4, 2, 2, 1)$ and $\mu = (2, 1)$ are partitions of $n = 9$ and $n' = 3$ respectively. We write $\lambda \vdash n$ and $\mu \vdash n'$, $|\lambda| = n$ and $|\mu| = n'$. The *Ferrers Diagram* F^λ associated to a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_p)$ consists of $|\lambda| = n$ boxes, arranged in $\ell(\lambda) = p$ left-justified rows of lengths λ_1, λ_2, ..., λ_p. Rows in F^λ can be oriented downwards or upwards. F^λ is called the shape of λ. If F^λ contains F^μ, then the *skew diagram* (or skew partition) λ/μ is the one obtained from F^λ by deleting F^μ. A *semi-standard Young tableau* ($SSYT^{\,\lambda}$) is a numbering of the boxes of F^λ with entries from $\{1, 2, ..., n\}$, weakly increasing across rows and strictly increasing up (or down) columns. A tableau is *standard* ($SYT^{\,\lambda}$) if all its entries are different. Skew tableaux are defined in an analogous way. For example, for $\lambda = (4, 2, 2, 1)$ and $\mu = (2, 1)$, the Ferrers diagram $F^{\,\lambda}$, the skew Ferrers diagram λ/μ, a semi-standard tableau and a standard tableau both of shape λ, and a standard skew tableau of shape λ/μ are as follows:



Schur functions, Kostka numbers and Littlewood-Richardson coefficients

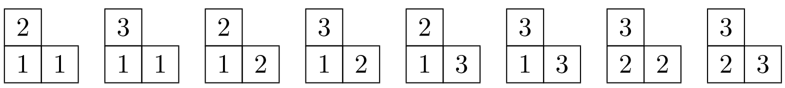
A *Symmetric Function* is a function which is symmetric or invariant under permutation of its variables.

$$f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) = f(x_1, x_2, ..., x_n) \tag{1}$$

where σ is any permutation of the symmetric group $\mathfrak{S}_\mathfrak{n}$. For example let us consider the function $f(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3$. If we express $f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ for any permutation σ in $\mathfrak{S}_3$, we get back the expression of $f(x_1, x_2, x_3)$. So f is an example of symmetric function on 3 commutative variables. One can read [7] for an introduction to symmetric functions. The set of all symmetric functions on commutative variables is an algebra spanned by many known bases one of which is the collection of Schur functions indexed by partitions of all integers. For $\lambda \vdash n$, the *Schur function* s_λ is the symmetric function defined as:

$$s_\lambda(\mathsf{x}) = \sum_{T \,\in\, SSYT(\lambda)} x_1^{m_1}\, x_2^{m_2} \,...\, x_n^{m_n} \tag{2}$$

where $SSYT(\lambda)$ is the set of all semi-standard tableaux of shape λ, m_i is the number of entries equal to i in T for $i = 1, 2, ..., n$. For example, for $\mu = (2, 1)$ there are 8 semi-standard tableaux of shape μ:



So the Schur function $s_\mu(\mathsf{x})$ is the symmetric function defined as:

$$s_{21}(\mathsf{x}) = x_1^2\, x_2 + x_1^2\, x_3 + x_1\, x_2^2 + 2\, x_1\, x_2\, x_3 + x_1\, x_3^2 + x_2^2\, x_3 + x_2\, x_3^2$$

One will get it this way using MuPAD-Combinat: first create an instance of the algebra of symmetric functions:

```
>> sym := examples::SymmetricFunctions():
```

then expand the corresponding Schur function on an alphabet of n variables:

```
>> alphabet := [x1, x2, x3]:
>> expand(sym::s([2,1])(alphabet));
x1^2 x2 + x1^2 x3 + x1 x2^2 + 2 x1 x2 x3 + x1 x3^2
+ x2^2 x3 + x2 x3^2
```

Littlewood-Richardson coefficients (L-R. coefs.) $c^{\,\nu}_{\lambda\mu}$ are defined as the structure constants for the multiplication in the basis of Schur functions. So if $\lambda \vdash n$ and $\mu \vdash m$:

$$s_\lambda\, s_\mu = \sum_{\nu\,\vdash\, n+m} c^{\,\nu}_{\lambda\,\mu}\, s_\nu \tag{3}$$

For example, it may be a little bit hard to compute it by hand, but one will find that:

$$s_{21}\, s_{21} = s_{42} + s_{411} + s_{33} + 2\, s_{321} + s_{3111} + s_{222} + s_{2211}$$

Thus $c^{\,42}_{21,\,21} = 1$ and $c^{\,321}_{21,\,21} = 2$.
The product of two symmetric functions, in the basis of Schur functions is performed this way using MuPAD-Combinat:

```
>> f := sym::s([2,1]) * sym::s([2,1]);
s[3,1,1,1] + s[4,1,1] + 2 s[3,2,1] + s[2,2,2] + s[2,2,1,1]
+ s[4,2] + s[3,3]
```

One may also want to express this product in another basis of the algebra of symmetric functions, for instance the basis of monomial symmetric functions:

```
>> sym::m(f);
2 m[3,1,1] + 20 m[1,1,1,1,1] + 9 m[2,1,1,1] + 4 m[2,2,1] + m[3,2]
```

Use `sym::e(f)`, `sym::h(f)` and `sym::p(f)` respectively to convert a symmetric function into its representation in the basis of elementary symmetric functions, complete symmetric functions or power sums, see [7] for definitions. One can also use lrcalc (Littlewood-Richardson Calculator), a package of C and $Maple$ programs, designed by A. S Buch [3] for computing L-R. coefs. We have interfaced it with MuPAD-Combinat.

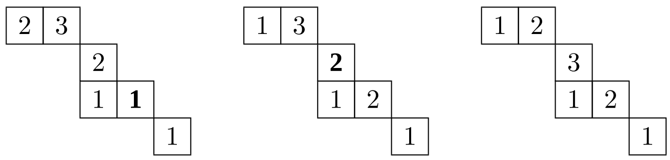
```
>> lrcalc::mult([2,1],[2,1]);
   [ [1, [3,3]], [1, [4,2]], [1, [2,2,1,1]], [1, [2,2,2]],
     [2, [3,2,1]], [1, [4,1,1]], [1, [3,1,1,1]] ]

>> L := [2,1]: M := L:
>> lrcalc::lrcoef([4,2],L,M), lrcalc::lrcoef([3,2,1],L,M);
                                   1, 2
```

L-R. coefs can be combinatorially computed by enumerating some combinatorial objects called *Yamanouchi words*. A right (respectively left) Yamanouchi word on a completely ordered alphabet, for instance $\{1, 2, \cdots\}$, is a word w such that any right (respectively left) factor of w contains more entries i than $(i+1)$. For example, the word

$w = 2322131211$ is a right Yamanouchi one. It's valuation is $eval(w) = (4, 4, 2)$. Yamanouchi words help compute L-R. coefs. According to a well known result named the *Littlewood-Richardson rule* [2], $c^{\nu}_{\lambda\,\mu}$ is the number of skew tableaux of shape ν/λ and valuation μ, and whose row readings are Yamanouchi words. All those constructions are based on a simple algorithm know as the Robinson-Schensted correspondence [2]. For example, let us consider the three partitions $\lambda = (5, 2, 2)$, $\mu = (3, 2, 1)$ and $\nu = (6, 4, 3, 2)$. The Yamanouchi skew tableaux of shape μ/λ and valuation μ are the following:

```
>> yamanouchi::list([[6,4,3,2],[5,2,2]],[3,2,1]);
```

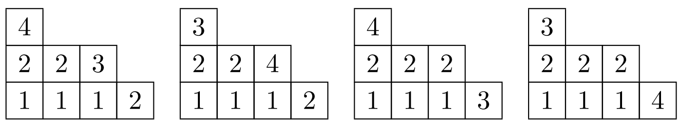


Thus $c^{6432}_{522,\,321} = 3$. The Yamanouchi tabeaux listed correspond to the following Yamanouchi words:

```
>> map(%,skewTableaux::toWord);
[[2,3,2,1,1,1], [1,3,2,1,2,1], [1,2,3,1,2,1]]
```

Kostka Numbers $K_{\lambda\mu}$ are the number of distinctly labeled semi-standard Young tableaux $(SSYT^{\,\lambda})$ of shape F^{λ} and weight μ, that is to say with μ_i entries i, for $i = 1, 2, ..., \ell\,(\mu)$. For example, if $\lambda = (4, 3, 1)$ and $\mu = (3, 3, 1, 1)$ then there are 4 semi-standard Young tableaux $(SSYT^{\,431})$ of weight $(3, 3, 1, 1)$. Thus $K_{431,\,3311} = 4$. Use the following commands to list those tableaux, or count them:

```
>> tableaux::list([4,3,1],[3,3,1,1]);
```



```
>> tableaux::kostka([4,3,1],[3,3,1,1]);
                     4
```

Kostka Numbers happen to be special kinds of Littlewood-Richardson coefficients [9]. So using the same technics as for the latter, one can compute the first. Most of their applications and utilizations are found in groups representation theory: Littlewood-Richardson coefficients govern the decomposition of tensor products of irreducible representations of the General Linear Group $GL(n)$ in accordance with the formula:

$$V^{\lambda} \otimes V^{\mu} = \sum_{\nu} c^{\,\nu}_{\lambda\,\mu}\, V^{\,\nu} \qquad (4)$$

They are also useful in computing induction product of modules over the symmetric group. See for example [9] for more details. So computing those numbers remains of great interest in algebraic combinatorics.

2. A problem finding polynomials to count lattice points in certain convex polytopes

Recall that L-R. coefs $c^{\,\nu}_{\lambda\mu}$ are defined as the structure constants for the multiplication in the basis of Schur functions (3). The problem we are presenting here is related to computing those numbers.

2.1. Problem nature

A wide variety of topics in pure and applied mathematics involve the problem of counting the number of lattice points inside a convex polytope of the form: $P = \{x\,:\, Ax \leq b,\ x \geq 0\}$, where A is an integral matrix, and b an integral vector. We are interested in a family of counting functions of the form: $\phi_A(N) = \{x\,:\, Ax \leq Nv,\, x \geq 0,\ x \text{ integral}\}$. That are functions which count the number of lattice points inside convex polytopes given in terms of a fixed matrice A and a right-hand-side vector b that is changing as a single-parameter dilatation of a fixed initial vector v. Geometrically, this is interpreted as dilating the associated polytope by a positive integer factor, while leaving the angles and proportions fixed. We first encountered this type of polytopes while interested in the computation of stretched Kostka numbers $K_{N\lambda,N\mu}$, and of stretched L-R. coefs $c^{\,N\nu}_{N\lambda,\,N\mu}$, that is those associated to fixed partitions λ, μ and ν all scaled by an integer N. Il is known ([9] and [5]) that those coefficients have a polynomial growth with respect to the dilatation factor $N \in \mathbb{N}$:

$$c^{\,N\nu}_{N\lambda,\,N\mu} = p^{\,\nu}_{\lambda\,\mu}(N) \qquad \text{with} \qquad p^{\,\nu}_{\lambda\,\mu}(0) = 1 \tag{5}$$

It is also possible to determine an upper bound ($maxDeg$) of the value of the degree of the polynomial p [10]. Given three partitions λ, μ and ν, the L-R. coef $c^{\,\nu}_{\lambda\,\mu}$ can be computed considering a model known as the hive model [9]. It consists in building a system of inequalities defining a convex polytope $P = \{x\,:\, Ax \leq v,\ x \geq 0,\ x \text{ integral}\}$, where A and v are entirely determined by λ, μ and ν. Then for any $N \in \mathbb{N}$, the stretched L-R. coef $c^{\,N\nu}_{N\lambda,\,N\mu}$ is the number of lattice points in the dilated polytope $NP = \{x\,:\, Ax \leq Nv,\, x \geq 0,\ x \text{ integral}\}$.

Given $maxDeg$, the upper bound of the value of the degree of the polynomial p; since $p^{\,\nu}_{\lambda\,\mu}(0) = 1$, the set of values $\{\,\phi_A(N),\ N = 1\,..\,maxDeg\,\}$ is sufficient to find $p^{\,\nu}_{\lambda\,\mu}$ by interpolation. We can achieve the determination of this set with the help (of a predefined subset) of the available computers of the Local Area Network: that is distributed computation. The step-by-step process is described in [8]. In the example bellow:

```
Lambda: (7,6,5,4) ; Mu: (7,7,7,4) ; Nu: (12,8,8,7,6,4,2)
Max. Deg:   8
Dilat.(N):  0    1    2     3     4      5      6      7      8
Coef.    :  1   12   62   212   567   1288   2604   4824   8349
P1(N): 11N + 1
P2(N): 39/2 N^2 - 17/2 N + 1
P3(N): 61/6 N^3 - 11 N^2 + 71/6 N + 1
P4(N): 11/6 N^4 - 5/6 N^3 + 55/6 N^2 + 5/6 N + 1
P5(N): 1/30 (N+3)(N+2)(N+1)(3N^2+7N+5) = P6(N) = P7(N) = P8(N)
Total number of successive identical polynomials: 4
Real. Deg:  5
```

are given three partitions λ, μ and ν, the predicted maximal degree ($maxDeg$) of the polynomial, a list of values of the sought-after polynomial, and a set of polynomials obtained by successive interpolations with k values for $k \leq maxDeg$. To get this output:

```
>> lrPol::([7,6,5,4],[7,7,7,4],[12,8,8,7,6,4,2]);
```

2.2. Some experimental results

Below is a table showing some of the computations we've carried out. Those examples were arbitrarily selected, and they suggest that the set of polynomials obtained by interpolation is often stationary. This means that for a given triple (λ, μ, ν), there is an integer i_0 probably depending on λ, μ and ν, such that for any $i \geq i_0$ the equality $P_i{}_{\lambda\,\mu}^{\,\nu} = P_{i_0}{}_{\lambda\,\mu}^{\,\nu}$ holds. So we usually don't need to interpolate the theoretical maximum number of points to get the final polynomial. In other words, the polynomial $P_{\lambda\,\mu}^{\,\nu}$ usually has lower degree than the maximum degree predicted.

n°	partition (λ)	partition (μ)	partition (ν)	**max degree**	**real degree**
1	9,7,3	9,9,3,2	10,9,9,8,6	4	1
4	6,5,2,2	5,5,3,2,2,2,1	8,8,7,7,2,2,1	5	0
5	9,8,3,3	10,7,5,3	10,10,8,8,7,5	5	2
6	11,10,8,5	20,17,3	26,25,8,8,5,2	5	4
7	9,5,3,3,3	7,6,5,4,3	10,10,8,8,7,5	5	5
13	7,6,5,4	7,7,7,4	12,8,8,7,6,4,2	8	5
16	11,10,8,4,2	8,7,6,5,2	18,17,15,7,4,2	10	2
18	5,4,4,3,3,2,1	9,7,3,3,2,2,1	10,9,8,7,6,5,4	11	0
20	5,5,3,2,1,1	6,6,4,2,1	6,6,6,5,5,3,3,2	12	3
21	5,5,3,2,1,1	6,6,4,2,1	6,6,6,5,5,3,2,2,1	14	5

An entry in the table gives for a triple of partitions λ, μ and ν the maximal degree of the polynomial counting lattice points in the corresponding dilated polytopes, and the rank from which the interpolations give the same polynomial. For the technics used to construct those polytopes, see [9] or [8].

2.3. One of the remaining questions

The formulation of the problem is completely independent of its origin. One can forget about partitions, Kostka numbers and Littlewood-Richardson coefficients, and just remind that: we are looking for a polynomial f with $deg(f) \leq d$ and $f(0) = 1$; $f(N)$ counts lattice points in a certain convex polytope, dilated by N; we need at most d values of f to interpolate and find f; we are receiving values of $f(N)$ in cascade, $1 \leq N \leq d$; we interpolate each time we receive new values, using all the already available values. One of the remaining questions is then the following:

should we decide to stop computations as soon as two consecutive polynomials obtained from two consecutive interpolations are equal.

In other words,

if the lattice points in the dilatations of a given polytope are counted by $f(N)$ where N is the dilatation factor and f a polynomial having degree d, and if there exists a family $(f_0 = 1, f_1, \cdots, f_d = f)$ of polynomials such that f_i counts lattice points in this polytope dilated by factors N for $0 \leq N \leq i$, then can one have $f_{i_0} = f_{i_0+1}$ and $f_{i_0+1} \neq f_{i_0+2}$ for a certain integer i_0.

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The author is a PhD student in theoretical computer science at the Université de Marne-la-vallée, France. He is interested in algebraic combinatorics and its applications: generalizations of the Schensted correspondence and Littlewood-Richardson rule for various combinatorial objects like binary search trees. Distributed computations, graphs duality and construction of Hopf algebras.

A. Software presentation: MuPAD-Combinat

MuPAD-Combinat is an open-source algebraic combinatorics package for the computer algebra system MuPAD. Its main purpose is to provide an extensible toolbox for computer exploration, and foster code sharing between researchers in this area. The development started in spring 2001, and the package currently contains functions to deal with usual combinatorial classes (partitions, tableaux, graphs, trees, decomposable classes, ...), Schubert polynomials, characters of the symmetric group, and weighted automata. It supplies the user with tools for constructing new combinatorial classes and combinatorial (Hopf) algebras. As an application, it provides some well-known combinatorial Hopf algebras like the algebra of symmetric functions and many generalizations. There is also some preliminary support for combinatorial Lie algebras and operads. The core of the package is integrated in the official library of MuPAD since version 2.5.0.

MuPAD-Combinat is freely available for download and its documentation is also available in *dvi*, *pdf* and *html* formats, at `http://mupad-combinat.sourceforge.net`.