Identification of pointwise sources and small size flaws via the reciprocity gap principle; Stability Estimates

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RÉSUMÉ. Un problème inverse d'identification de point sources dans un domaine du plan à partir des données sur la frontière est considéré. Un résultat local de stabilité Lipschitzienne est donné.
Un procédé efficace et très élémentaire d'identification est exécuté et appliqué à des inclusions de petites tailles et de faible conductivité.

ABSTRACT. The inverse problem of locating point wise sources in 2-D domain from over specified boundary data is considered. A local Lipschitz stability result is given.
An efficient and very elementary identification process is performed and applied to small size conductivity defaults location.

MOTS-CLÉS : Point sources, équation de Laplace, fonction écart à la réciprocité, Problème inverse.

KEYWORDS : Source points; Laplace equation; Reciprocity gap functional; Inverse problem.

In this presentation we study the issues of stability and identification in the problem of determining the locations and strengths of point sources for Poisson's equation in 2-D situations. When studying this inverse problem, we have in mind two applications. The first one is borrowed from the bio-engineering communauty it concerns the location of epileptic foci in the humain brain. This problem is known as the inverse electro-encephalography problem. Epilepsy foci are usually modelled by point wise current dipoles. The second motivation arises from the need to detect smooth small inclusions scattered in planer matrix phase with a known background conductivity. In all this paper we will refer to the following model problem :

$$(P)\left\{\begin{array}{lcll} -\Delta u & = & \displaystyle\sum_{j=1}^{m_1}\lambda_j\,\delta_{S_j}+\sum_{j=1}^{m_2}\vec{P}_j\cdot\nabla\delta_{S'_j} & in\ \Omega \\ u & = & f & on\ \Gamma \\ \frac{\partial u}{\partial \nu} & = & \varphi & on\ \Gamma \end{array}\right. \qquad [1]$$

Where Ω is a bounded domain of $\mathbb{R}^2$ with sufficiently smooth boundary $\Gamma,\ S_i,\ \lambda_i$ are the monopolar points sources and their intensities, $S'_i,\ \vec{P}_i$ are the dipolar points sources and their moments, ν is the outer unit normal of the surface $\Gamma,\ f \in H^{\frac{1}{2}}(\Gamma)$ and $\varphi \in H^{-\frac{1}{2}}(\Gamma)$. An uniqueness result is established by El Badia and T.Ha-Duong in [3]. Our inverse problem is therefore, defined by the over determined boundary data (f,φ) and it concerns the locations the point sources S_i and the recovering of their intensities. Let us point out that very little is known mathematically about stability of such a problem. The last issue in studying such an inverse problem is dedicated to the inversion process. We are concerned by the number of sources, their locations, their intensities and moments. The algorithm in [3] is an algebraic method based on the reciprocity gap principle $\mathcal{R}$, introduced in [1] for the inverse crack problem.

1. Stability results

1.1. The point wise sources case

The goal of this section is to study the stability of the inverse identification problems under consideration, that is, roughly speaking to study if small perturbation in measurements lead to points with their intensities in the vicinity of the original points sources and their original intensities. We assume that both the flux and the electrical current are available on the boundary Γ. To prove local Lipschitz stability, we use the concept of reciprocity gap functional which is defined by :

$$\mathcal{R}(v)=\langle\varphi,v\rangle_{H^{\frac{1}{2}}(\Gamma),H^{-\frac{1}{2}}(\Gamma)}-\langle f,\frac{\partial v}{\partial n}\rangle_{H^{\frac{1}{2}}(\Gamma),H^{-\frac{1}{2}}(\Gamma)}$$

where $v\in H(\Omega)=\{v \in H^1(\Omega),\ \Delta v=0\}$. Using the second Green formula, it is straight forward :

$$\mathcal{R}(v)=\sum_{j=1}^{m}\lambda_j\,v(S_j) \quad \forall\, v \in H(\Omega) \qquad [2]$$

Where the same notation is used for a point source S_j resp(T_j) and its complex affixe, we note that the corresponding intensities λ_j resp(μ_j) can now be authorized to be complex.

We consider the two problems $P_i \quad (i=1,2)$ and let us estimate the error $d(A,B)$ where $d(A,B) = \min_{\sigma} \sum_{j=1}^{m} |S_j - T_{\sigma(j)}| + |\lambda_j - \mu_{\sigma(j)}|$ and $\sigma \ \in \ S(n), \ S(n)$ being the set of the permutations of $\{1,2,...,m\}$.

$$(P_1)\begin{cases} -\Delta u &= \sum_{j=1}^{m} \lambda_j \ \delta_{S_j} & in \ \Omega \\ u &= f_1 & on \ \Gamma \\ \frac{\partial u}{\partial \nu} &= \varphi & on \ \Gamma \end{cases} \qquad [3]$$

and (P_2) is the Laplace problem when $\lambda_j, \ S_j$ and f_1 are replaced by $\mu_j, \ T_j$ and f_2. where $f_1, \ f_2$ are in $H^{\frac{1}{2}}(\Gamma)$ and φ in $H^{-\frac{1}{2}}(\Gamma)$. We consider the positive numbers M_1 ,M_2 and R, we assume that the following conditions hold true :

$$|S_j - S_k| \geq M_1, |T_j - T_k| \geq M_1 \quad \forall \quad j,k = 1.2,...,m \qquad j \neq k \qquad [4]$$

$$|S_j - T_k| \geq M_1 \qquad \forall \quad j,k = 1.2,...,m \qquad j \neq k \qquad [5]$$

$$|\lambda_j| \geq M_2, \quad |\mu_j| \geq M_2 \quad \forall \quad j = 1,2,...,m \qquad [6]$$

$$|z| \leq R \quad \forall \quad z \in \quad \Omega \qquad [7]$$

Theorem 1 *. Assume that the configurations $A = \{\lambda_j, S_j\}_{1 \leq j \leq m}$ and $B = \{\mu_j, T_j\}_{1 \ \leq \ j \ \leq \ m}$ satisfy the conditions (4), (5),(6) and (7), then we have :*

$$d(A,B) \leq c \quad \|f_1 - f_2\|_{L^2(\Gamma)}$$

where c is a constant depending only on $M_1, M_2, \ \Omega$ and R where R is a bound on the size of Ω.

Remark The same result is valid for dipolar sources.

2. Numerical experiments

2.1. Point wise sources : the numerical procedure

Our numerical process is based on the work of El-Badia and Ha-Duong in [3], for the readers convenience we recall the main result of [3]. We denote for $j \in \ \mathbb{N}$, $\alpha_j = \mathcal{R}(z^j), \ z^j, \ z = x + iy$ are tests harmonic functions,

$$\mu_j = \begin{pmatrix} \alpha_j \\ \alpha_{j+1} \\ . \\ . \\ . \\ \alpha_{j+M-1} \end{pmatrix} \in \ C^M, \quad A_j = \begin{pmatrix} S_1^j & S_2^j & . \ . \ . & S_m^j \\ S_1^{j+1} & S_2^{j+1} & . \ . \ . & S_m^{j+1} \\ S_1^2 & S_2^2 & . \ . \ . & S_m^2 \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ S_1^{M+j-1} & S_2^{m-1} & . \ . \ . & S_m^{M+j-1} \end{pmatrix} \in \ \mathcal{M}_{M \times m}(C)$$

where M is an upper bound known of m and $\Lambda = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ . \\ . \\ . \\ \lambda_m \end{pmatrix} \in C^m$.

Lemma 2 *:[3]*
The number of monopolar source points is the rank of the vectors $\{\mu_0, \mu_1, ..., \mu_{M-1}\}$.

Lemma 3 *:[3]*
-The affixes $S_j,\ j = 1, ..., m$ *of the source points are the eigenvalues of* T *where* T *is the matrix defined by :*$T(\mu_j) = \mu_{j+1}\ j = 0, ..., m-1$.
-The intensities $\lambda_j,\ j = 1, ..., m$ *are solution of the system :* $\mu_0 = A_0\ \Lambda$.

Remark :
For identifying the dipolar source points and refereing to [3], we use the same algorithm taking $\beta_j = \frac{\alpha_j}{j}$ in the place of α_j for $j = 1, ..., m$.
For numerical approach, we remark that for $n = 1, ..., M$ the matrix :

$$(\mu_0\mu_1...\mu_{n-1}) = \begin{pmatrix} \lambda_1 & \lambda_2 & . & . & . & \lambda_m \\ \lambda_1 S_1 & \lambda_2 S_2 & . & . & . & \lambda_m S_m \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ \lambda_1 S_1^{n-1} & \lambda_2 S_2^{n-1} & . & . & . & \lambda_m S_m^{n-1} \end{pmatrix} \begin{pmatrix} 1 & S_1 & . & . & . & S_1^{n-1} \\ 1 & S_2 & . & . & . & S_2^{n-1} \\ . & & & & & \\ . & & & & & \\ . & & & & & \\ 1 & S_m & . & . & . & S_m^{n-1} \end{pmatrix}$$

where $\mu_j = \begin{pmatrix} \alpha_j \\ \alpha_{j+1} \\ . \\ . \\ . \\ \alpha_{j+n-1} \end{pmatrix}$ $0 \leq j \leq n-1$ Then $D_n = det(\mu_0, \mu_1, ..., \mu_{n-1}) =$
$\lambda_1\lambda_2...\lambda_n \prod_{i<j}(S_i - S_j)^2$. if $m = n$.
If $n > m$, we can reduce the matter to the case $n = m$ by letting $\lambda_{m+1} = ... = \lambda_n = 0$ and $D_n = 0$. In particular $|D_n| \geq M_2^n M_1^{n^2-n}$. Refereing to [6] the number of the source points is the integer m for which $|D_m| > \frac{1}{2} M_2^m M_1^{m^2-m}$ and $|D_i| \leq \frac{1}{2} M_2^i M_1^{i^2-i}$ $i = m+1, ..., M$. Where M_1 and M_2 given in (6),(7). The integer m is the number of point sources.

2.2. Numerical trials :

For simplicity, all the tests of this section are performed for the 2-D unit disc. The boundary data of the discussed cases(i.e. the measurements) are generated by the fundamental solution of the problem (1). Before testing the method with imposed different levels of noise on the boundary data, we will begin with a noise free case to point out that the accuracy of the evaluation of $\mathcal{R}(z^j)$ is a crûrcial point in the inversion process to avoid that fictitious point sources show up. Another way to filter these undesirable points is to impose to M to be equal to the apriori unknown number m of source points. This will be possible with the method developed here after and which will be a good tool to smooth the solution for the noisy data problem. This aspect will be clarified in the subsection (2.2.2).

2.2.1. Noise free case :

The results of this subsection are performed to show the efficiency of the above method for a noise free case. However, even for detecting only one source point, for a little number of integration nodes on the boundary, the value of the reciprocity gap functional at z^j have not a good approximation, and some other very small intensity source points (fictitious points) appears. These points can be filtered by imposing an admissibility test. All the recovered sources which are out of the domain or having an intensity less than a fixed tolerence epsilon value are eliminated. However, apriori selection of such an ϵ is not set up. Notice that no problem arises when the number source points is apriori known. In the case of a low number of measurements we can obtain a satisfying result.In the method described in the subsection (2.1) the number of source point is assumed unknown and the elimination of parasite sources is done by increasing the measurements. Paradoxically, the method which we will describe in the next section will allow to estimate the number of source points in a noisy data case and not in a noisy free case one. Let's observe that with a fixed number nd of data on the boundary, we can resolve n times little approximated problems by considering for each one $\frac{nd}{n}$ data points number. This sampling of the measured data will give us n less accurate solutions with areas containing fictitious points and other containing admissible ones (Fig. 1).

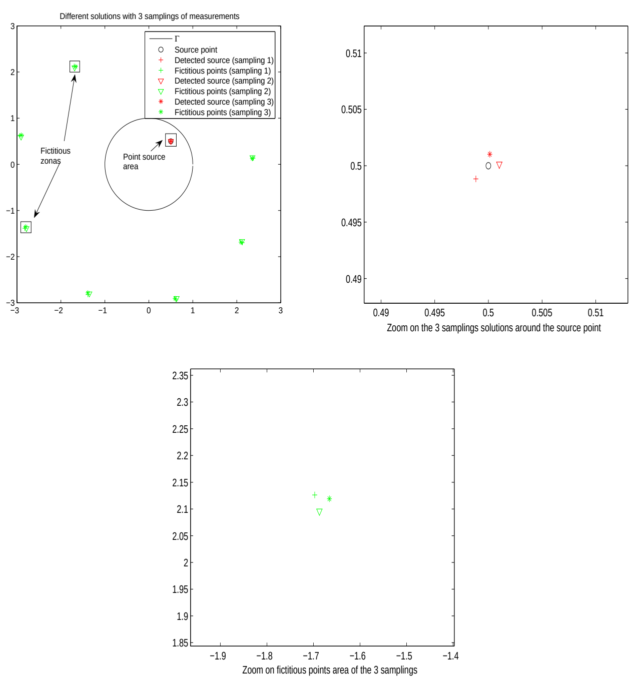

Figure 1.

2.2.2. Noisy data case :

With noisy data, the solution given by the procedure described in (2.1) section loose of their accuracy and we can note that the algorithm is not robust particularly in presence of relatively great number of source points. In this case we can observe that the solution is composed by fictitious points inside and outside the domain and five source points are detected instead of four. This result illustrate that the admissibility test method which impose to the solution to be located in the domain with an intensity greater then $\frac{1}{10}$ is not efficient. By fractionating the measurements in some samplings, the superposition of the identified points obtained by all the samplings give the zonas, around the source points, with high density of identified points and zonas formed with some isolated points Fig2.Obviously four dense zonas are detected. Recalculating the solution with all the data by imposing to the number of source point, and then to the order of the matrix T, to be equal to the dense zone number, we obtain regarding the noise level a good result of locations and intensities, Fig.3.

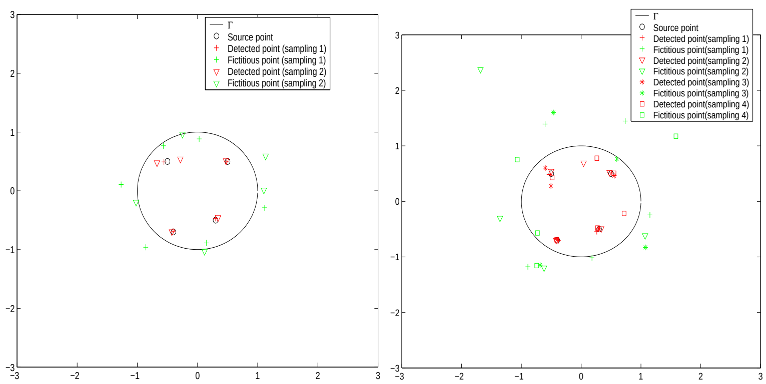

Figure 2. *Zonas obtained by 2 and 4 sampling of the measurements.*

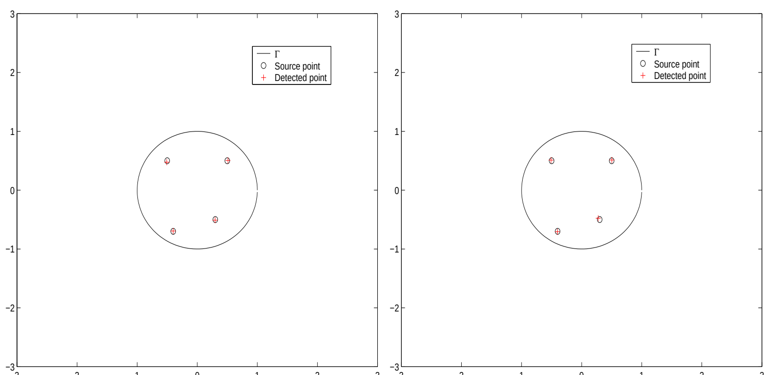

Figure 3. *Identification when the order of the matrix T is equal to the dense zone number, noise=5%, 10%*

We add to the over specified boundary data a random noise with respect to the maximum value of the measured data. The method is stable when the domain contains a small number of monopolar sources. In fact, we obtain good results for perturbed data with an additive random noise. The identification process deteriorates as the noise increases.

2.2.3. Small size inclusions

Our second application is devoted to the detect of inclusions of small size. We have in mind the detection of mines. Mines are considered to be small with respect to the prospected area and to have a significantly higher (metal) or lower (plastic) conductivity than the surrounding soil. Taking advantage of the smallness of the inclusions, asymptotic analysis was developed in order to design reconstruction algorithms. Let Ω be a conductor with conductivity 1 and assume that Ω contains conductivity imperfection D_j of he form $D_\epsilon^j = z_j + \epsilon\ B_j,\ j = 1, ..., m$, where B_j are reference domains with Lipshitz boundaries and ϵ is a small number. The points $z_j \in \Omega$ determine the location of the imperfections whereas the domains $B_j \subset \mathbb{R}^2$ which describe their relative shapes, are bounded, smooth, strictly star-shaped, and contain the origin 0. The parameter ϵ measures the common order of magnitude of the inclusions diameter : it is sufficiently small, so the inclusions are distant enough from the boundary $\partial\Omega$. Suppose that the conductivity of D_j is k_j for $j = 1, ..., m$. If we apply a current g on $\partial\Omega$, then the voltage potential u_ϵ in the presence of D_ϵ^j the voltage potential u_ϵ is the H^1-solution to

$$(P_2)\begin{cases} \nabla.(\gamma_\epsilon)\nabla u_\epsilon & = & 0 & in\ \Omega \\ \frac{\partial u_\epsilon}{\partial \nu} & = & g & on\ \partial\Omega \\ \int_{\partial\Omega} u_\epsilon & = & 0 & \end{cases} \qquad [8]$$

Here ν denotes the unit outward normal to the domain Ω. The conductivity γ_ϵ is assumed to be equal to 1 in the safe part of the domain $\Omega \setminus \cup_{j=1}^m D_\epsilon^j$ while on the j^{th} inclusion D_ϵ^j, we have $\gamma_\epsilon = k_j$, for some constant $k_j \in \mathbb{R}_+^\star$. The perturbation of voltage potential due to the presence of small-separated inclusions $\cup_{j=1}^m D_\epsilon^j$ is given $z \in \partial\Omega$ by

$$u_\epsilon(z) - U(z) + 2\int_{\partial\Omega}(u_\epsilon(x) - U(x))\gamma\frac{\partial\Phi}{\partial\nu_x}(x,z)d\sigma_x = 2\epsilon^n\sum_{i=1}^m \gamma\frac{\gamma - k_i}{k_i}\nabla_x\Phi(z_i,z).M_i\nabla_x U(z_i) +$$

$O(\epsilon^{n+\frac{1}{2}})$. Where $\Phi(x,y) = -\frac{1}{2\pi\gamma}log|x-y|,\ U$ is the solution in the safe body and γ the background conductivity is constant,M_i is the polarisation matrix associated with the domain B_j. In two-dimensional case, with the expression $-\frac{1}{2\pi\gamma}log|x-y|$, the previous formula reads

$$u_\epsilon(z) - U(z) + \frac{1}{\pi}\int_{\partial\Omega}(u_\epsilon(x) - U(x))\gamma\frac{(z-x).\nu_x}{|x-z|^2}d\sigma_x = \epsilon^2\frac{1}{\pi}\sum_{i=1}^m \gamma\frac{\gamma - k_i}{k_i}\frac{(z-z_i).\nu_x}{|z-z_i|^2}.M_i\nabla_x U(z_i) +$$

$O(\epsilon^{2+\frac{1}{2}})$. This asymptotic expansion show that this inclusions when ϵ is small have behave like a dipolar sources. Now, given additional measurements of the potential on the boundary : $u_{\epsilon\setminus\partial\Omega} = \phi_\epsilon$, the inverse problem consists in recovering the inclusions parameters k_j, y_j, B_j, and ϵ. In mine detection, however, as in some other applications, the informations of real interest are the positions y_j of the inclusions. We now want to detect smooth inclusions of small size by the same algorithm described in the beginning of this section. For our numerical tests, the applied boundary current flux is $\varphi(\theta) = cos(\theta)$. The domain is the unit ball of conductivity 1, the small disks are the inclusions of conductivities 0.01 with diameter $\epsilon = 0.04$, which generates synthetically the data (i.e. the potential

u on the boundary Γ) by finite element under the soft work MATLAB. In figure 4 the inclusions are represented by a small circle and the dipoles detected by $\star$.

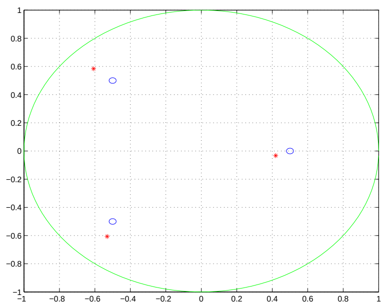

Figure 4. *Identification of inclusions.*

3. Conclusion

The main result of this work concerns the stability issue. It has been proved a local stability result in the inverse problem of locating pointwise conductivity defaults by very elementary tools. Such an inverse problem originate from biomedical applications, environnement such that detection of pollution sources. Numerical experiments based on an algebraic method [3], [6] have been performed. The main issue to be explored concerns the study of the realistic situation of incomplete boundary data (i.e. the overspecified data is available on a strict subset of the boundary). One possible direction consists, as a first step, on reconstructing the missing data before running the recovering algorithm.

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