A dual algorithm for denoising and preserving edges in image processing

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ABSTRACT. The restoration is a primordial step on image processing. We propose a variational model which permits to simultanuously denoise and preserve edges on image. By the use of the duality theory we prove that the minimization problem can be solved by considering its dual. First we treat an existance and unicity results of solution of the dual problem, and then we describe the algorithm for computing the solution. Finally, we present some numerical results for images on one and two dimension.

RÉSUMÉ. La restauration est une étape primordiale en traitement d'images. On propose un modèle variationnel qui permet simultanément de débruiter et de préserver les contours dans l'image. Par usage de la théorie de la dualité on montre que le problème de minimisation peut être résolu en considérant son dual. On traite au début les résultats d'existance et d'unicité d'une solution du problème dual ensuite on décrit un algorithme pour calculer la solution. Enfin on présente quelques résultats numériques pour des images en dimension 1 et 2.

KEYWORDS : Denoising, edge preservation, duality

MOTS-CLÉS : Débruitage, préservation des contours, dualité

1. Introduction

Let us consider a numerical image defined by a function f mapping a rectangle $\Omega =]0, l[\times]0, L[$ into R, f is assumed to belong to $L^2(\Omega)$. This function is the one we are considering throughout this paper, and following ideas introduced by several authors (see for example O. Faugeras [2] or G. Aubert [1]), we shall introduce the following optimization problem:

$$(\mathcal{P}) \left\{ \begin{array}{ll} \text{Minimize} & J(v) = \dfrac{1}{2}\displaystyle\int_\Omega |f-v|^2 + \varepsilon \int_\Omega \sqrt{1+a|\nabla v|^2} \\ v \in BV(\Omega) & \end{array} \right. \qquad [1]$$

where a and ε are two parameters to be adjusted in order to satisfactory results and $BV(\Omega)$ is the space of functions of bounded variations [1] defined by :

$$BV(\Omega) = \left\{ u \in L^1(\Omega); \sup \int_\Omega u.\text{div}\varphi < +\infty; \varphi \in C_0^1(\Omega)^n, |\varphi|_{L^\infty(\Omega)} \leq 1 \right\}$$

Let us point out however that, for (locally) small values of $|\nabla v|$, this functional J behaves like :

$$J^0(v) = \frac{1}{2}\, ||v-f||^2_{L^2} + \frac{\varepsilon a}{2}\, ||\nabla v||^2_{L^2} \qquad [2]$$

which makes problem (1) a classical filtering process [5], eliminating the small perturbations on the gradient of f and the solution of minimization defined on the space $H_0^1(\Omega)$. On the contrary, if v contains large gradients - at least locally - the potential functional J behaves like the following one:

$$J^1(v) = \frac{1}{2}\, ||v-f||^2_{L^2} + \varepsilon\sqrt{a}\, ||\nabla v||_{L^1} \qquad [3]$$

In that case, the minimizers of J^1 are to be sought in a functional space larger than $H_0^1(\Omega)$, which is the space $BV(\Omega)$, the crucial feature of which is to allow discontinuities, and thus enable the description of shape delimitations [4] .

Thus the minimization problem (1) is expected to be efficient for denoising purposes as well as for preserving discontinuities. Our goal here is to solve it by substituting a simpler one actually obtained by dualization.

2. A dual model

Let v be a function in $BV(\Omega)$, set:

$$p = \nabla v \qquad [4]$$

Then (1) is turned into the following constrained minimization problem :

$$(\mathcal{R}) \begin{cases} \min\limits_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \dfrac{1}{2}\displaystyle\int_\Omega |f-v|^2 + \varepsilon \int_\Omega \sqrt{1+a\,|q|^2} \\ q = \nabla v \end{cases} \qquad [5]$$

We use the Lagrangien in such a way that the unknown u is obtained as the first argument of the saddle point of the Lagrangien.

The Lagrangian $\mathcal{L}$ associated to problem (5) is given by (see [3]) :

$$\mathcal{L}(v,q,\mu) \quad = \quad \frac{1}{2}\int_\Omega |f-v|^2 + \varepsilon \int_\Omega \sqrt{1+a\,|q|^2} + \int_\Omega (\nabla v - q)\mu \qquad [6]$$

where the Lagrange multiplier μ belongs to $H_0(\text{div},\Omega)$:

$$H_0(\text{div},\Omega) = \left\{\lambda \in (L^2(\Omega))^2, \text{div}\lambda \in L^2(\Omega), \quad \lambda v = 0 \quad \text{on} \quad \partial\Omega\right\}$$

Thus, we have the following equivalence:

$$(1) \Longleftrightarrow \min_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \; \sup_{\mu\in H_0(\text{div},\Omega)} \mathcal{L}(v,q,\mu) \qquad [7]$$

the dual problem of which is :

$$\max_{\mu\in H_0(\text{div},\Omega)} \; \min_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \mathcal{L}(v,q,\mu) \qquad [8]$$

Once we solve the minimum problem with respect to (v,q)

$$\mathcal{L}(u,p,\mu) = \min_{(v,q)\in BV(\Omega)\times(L^2(\Omega))^2} \mathcal{L}(v,q,\mu)$$

Using the Euler equations, we get :

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial v} = 0 \quad \Rightarrow \quad u = f + \text{div}\mu \\ \dfrac{\partial \mathcal{L}}{\partial q} = 0 \quad \Rightarrow \quad p = \dfrac{\mu}{\sqrt{a}\sqrt{\varepsilon^2 a - |\,\mu\,|^2}} \end{cases}$$

By replacing u and p, the dual problem is turned into the following one :

$$(\mathcal{Q}) \left\{ \min_{\mu \in K} \; \frac{1}{2}\int_\Omega |\text{div}\mu|^2 + \int_\Omega f \text{div}\mu - \frac{1}{\sqrt{a}}\int_\Omega \sqrt{a\;\varepsilon^2 - |\mu|^2} \right. \qquad [9]$$

where

$$K = \left\{\mu \;/\; \mu \in H_0(\text{div}, \Omega), |\mu| \leq \varepsilon\,\sqrt{a}\right\}$$

3. Algorithm for the resolution of the dual problem

We will propose an algorithm for solving the dual problem (9).

First we change variables to get a nicer problem :

Let us set $\eta = \dfrac{\mu}{\sqrt{a}}$ and $z = \sqrt{\varepsilon^2 - \eta^2}$, then $\eta \in K_\varepsilon = \{\eta \in H_0(\text{div}, \Omega); |\eta| \leq \varepsilon\}$, and the dual problem becomes :

$$\min_{(\eta, z) \in \mathcal{C}} \left\{ \frac{a}{2}\int_\Omega |\text{div}\;\eta|^2 + \sqrt{a}\int_\Omega f\;\text{div}\;\eta - \int_\Omega z \right\} \qquad [10]$$

where

$$\mathcal{C} = \left\{(\eta, z) \in K_\varepsilon \times L^2(\Omega) \quad / \quad \eta^2 + z^2 = \varepsilon^2; z \geq 0\right\}$$

Two difficulties arise here:

1) The first one is that the constraint set $\mathcal{C}$ is not convex

2) The second one is related to the non uniqueness of η. Actually, uniqueness holds only up to a divergence free function.

To overcome the first of these problems, we have replaced the non convex set $\mathcal{C}$ by the convex one $\mathcal{D}$

$$\mathcal{D} = \{(\eta, z) \in H_0(\text{div}, \Omega) \times L^2(\Omega) \quad / \quad \eta^2 + z^2 \leq \varepsilon \quad ; \quad z \geq 0\}$$

and proved that minimizing on $\mathcal{D}$ is actually equivalent to minimizing on $\mathcal{C}$.

As for the second problem, we restore coercivity in (10) by adding an appropriate regularization term:

$$\min_{(\eta,z)\in D} \frac{a}{2}\int_\Omega |\text{div }\eta|^2 + \sqrt{a}\int_\Omega f\text{div }\eta + \frac{a\beta}{2}\int_\Omega |\eta|^2 - \int_\Omega z$$

Existence and uniqueness of a solution (η^β, z^β) thus hold in $\mathcal{C}$. Although only weak convergence of η^β is achieved, it can be proved that:

$$u^\beta = f + \sqrt{a}\text{ div }\eta^\beta \overset{L^2(\Omega)}{\longrightarrow} u = f + \sqrt{a}\text{ div }\eta^* when \beta \to 0$$

where η^* is the the dual problem solution having a minimal L^2 norm.

One should stress here that the restoration problem is only concerned with the divergence of the non regularized problem solutions, and that all of these solutions are equal up to a divergence free function. The above convergence result is therefore the very relevant one we need.
The algorithm that we use to solve this problem is the following :

Gradient algorithm with projection

- First guess (η_0, z_0)
- $k-$th iteration

$$(\eta_{k+1}, z_{k+1}) = \Pi_{\mathcal{D}}((\eta_k, z_k) - \rho_k \nabla \mathcal{H}_\beta(\eta_k, z_k))$$

- Stopping criterion : $||\eta_{k+1} - \eta_k||_{(L^\infty)^2} \leq \alpha$ (α is a given threshold)

with $\mathcal{H}_\beta(\eta, z) = \frac{a}{2}\int_\Omega |\text{div }\eta|^2 + \sqrt{a}\int_\Omega f\text{div }\eta + \frac{a\beta}{2}\int_\Omega |\eta|^2 - \int_\Omega z$

4. Numerical results

As for the approximation of the space $H_0(\text{div}, \Omega)$, the Raviart-Thomas finite elements have been used.
We illustrate the results of the regularization by using the problem of the minimization (1).

4.1. 1D Results

Our first experiment consists on considering the case where the observed image $f(x)$ is noisy and discontinuous and x defined on one dimensional space such as the segment $]0,1[$. We have the following results :

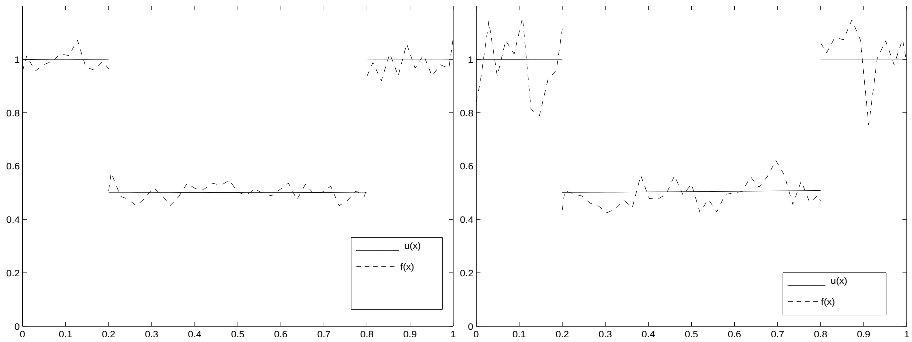

Figure 1. *Results of the restauration with the problem (1) when the noise is $\delta = 5\%$ in the first figure then $\delta = 10\%$ in the second one.*

Figure (1) shows that, applied in 1D situation, this algorithm denoises as expected, and in the same time restores discontinuities. We notice that the value of parameters of regularization was made in empiric way.

4.2. 2D Results

We observe that the solution of the restauration problem (1) is obtained by adding to the processed image the divergence of the solution of the dual problem (9) and which point out the discontinuities of the image, and we illustrate it by the following figures :

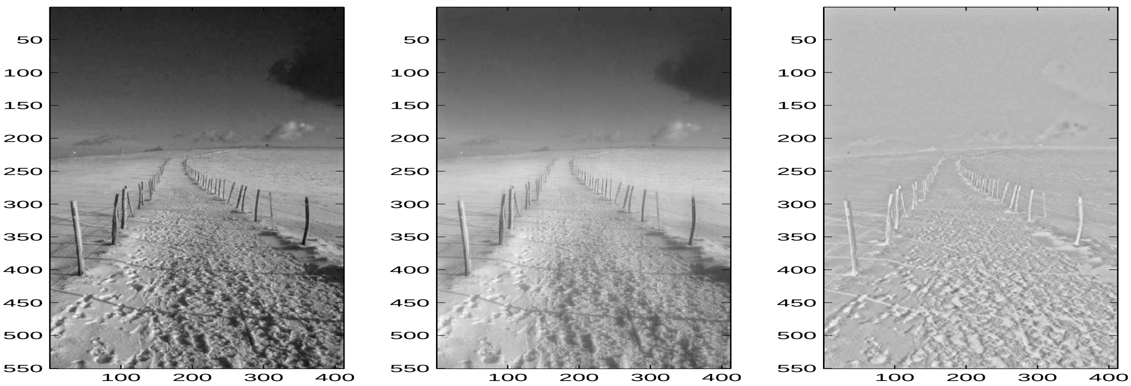

Figure 2.*Original image* *Processed image* *divλ*

We illustrate in the following the results of the restoration by using the problem (1) to noisy images :

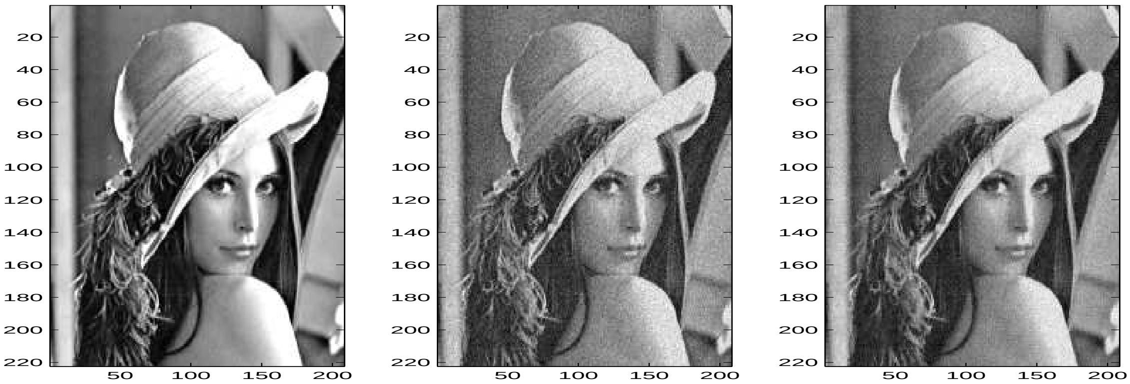

Figure 3. *Original image* *Noisy image (20%)* *Restored image*

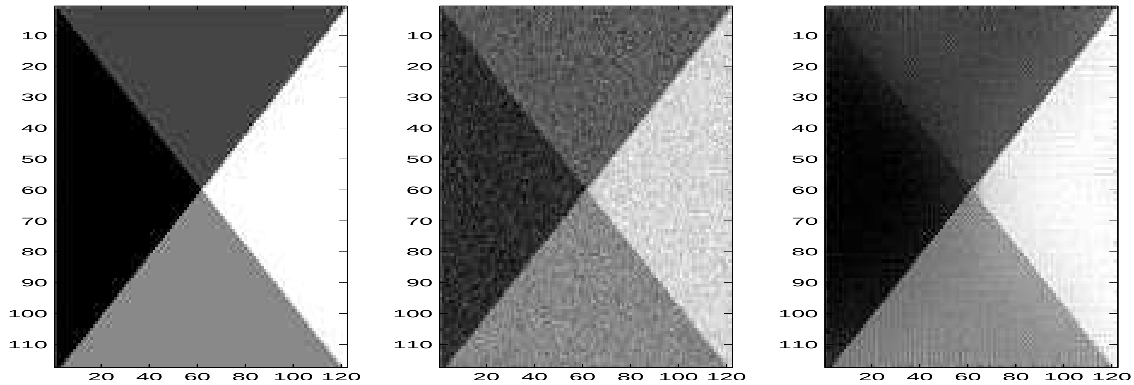

Figure 4. *Original image* *Noisy image (20%)* *Restored image*

Figures 3 and 4 shows some 2D experiments which lead to similar conclusions.

5. Conclusion

This paper presented an original method to resolve one problem of restoration of image with edge preservation and which difficulty consists on the discretization on the space of functions of bounded variations. The use of duality allow us, despite that the primal problem and the dual one are not equivalent (duality gap), to recover the solution of the restoration problem from the dual by the extremality relation.

6. Bibliography and biography

6.1. Bibliography

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6.2. Biography

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7. Acknowledgement

The authors are indebted to the SARIMA project which allowed Hela Sellami to spend 6 months stay in UNSA and CNAM in order to go ahead with the present work. The LAMSIN's researchers work is supported by the Tunisian Ministry for Research, Technology and Skills Development.