Dual mixed finite element methods of the elasticity problem with Lagrange multipliers

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Abstract: We study a dual mixed formulation of the elasticity system in a polygonal domain of the plane with mixed boundary conditions and its numerical approximation. The Neumann boundary conditions (or traction boundary condition)is imposed using a Lagrange multiplier corresponding to the trace of the displacement field. Moreover the strain tensor is introduced as a new unknown and its symmetry is relaxed, also by the use of a Lagrange multiplier (the rotation). The singular behavior of the solution requires to use refined meshes to restore optimal rates of convergence. Uniform error estimates in the Lamé coefficient$\lambda$are obtained for large λ.

Keywords: Mixed FEM, Lagrange multiplier, elasticity problem, inf-sup condition.

Résumé: Nous étudions une formulation duale mixte du problème de l'élasticité dans un domaine polygonal du plan avec des conditions au bord mixtes et son approximation numérique. La condition de Neumann est imposée en utilisant un multiplicateur de Lagrange qui est la trace du champ de déplacement. En outre le tenseur de contrainte est introduit comme nouvelle inconnue. Le comportement singulier de la solution nous amène à considérer des maillages raffinés pour avoir des taux de convergence optimaux. Des estimations d'erreur uniformes en terme du coefficient de Lamé λ sont obtenues pour de grandes valeurs de λ.

Mots clés: MEF mixte, multiplicateur de Lagrange, problème de l'élasticité, condition inf-sup.

1 Introduction

The analysis of classical finite element methods with Lagrange multiplier, originally developed in [1] has been considered for diverse problems, like the Laplace problem, the biharmonic equation or the Stokes system. On the other hand, the dual mixed finite element method (see [3, 13, 14]) has the advantage to introduce new unknwons like stresses and/or fluxes, quantities of physical interests, which are then computed directly with a good accurency, avoiding to use numerical postprocessing. Many papers are devoted to the elasticity system, let us quote [3, 6, 7]. For the elasticity system, this method has furthermore the advantage to avoid locking effect for large Lamé coefficient λ.

Recently Babuska and Gatica [2] have introduced a dual mixed finite element method for the Laplace equation with a Lagrange multiplier in order to impose nonhomogeneous Neumann boundary conditions.

Accordingly the goal of our paper is to extend the analysis made for the Laplace equation in [2] to the elasticity system. We furthermore want to take into account the singular behavior of the solution near the singular points of the domain by using refined meshes. Therefore contrary to [2], we do not use quasi-uniform meshes but use locally refined meshes. As a consequence we need to modify the norm of the approximation space in order to obtain a uniform discrete inf-sup condition. In [10, 11] the authors used a weighted mesh-dependent norm, we here prefer to use a standard L^2-norm . In comparison with the norm used in [2] and in [10, 11], our norm is more simple in a pratical point of view.

2 The dual mixed variational formulation

Let Ω be a simply connected domain of $\mathbb{R}^2$ with polygonal boundary Γ such that the interior angle at each corner lies in $(0, 2\pi)$. Let Γ_D and Γ_N be disjoint open subsets of Γ such that $|\Gamma_D| \neq 0$ and $|\Gamma_N| \neq 0$ and $\Gamma = \Gamma_D \cup \Gamma_N$ (the symbol $|\cdot|$ means here length).

In the static theory of linear isotropic elasticity, the equation satisfied by the

$$\text{displacement field u is} \quad -\operatorname{div}\sigma_s(u) = f \text{ in } \Omega, \tag{1}$$

where f represents the body force density, $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$ is the strain tensor,

$$\sigma_s(u) = 2\mu\varepsilon(u) + \lambda tr\varepsilon(u)\delta,$$

is the stress tensor, δ is the identity tensor, and finally μ, λ are the Lamé coefficients with $\mu \in [\mu_1, \mu_2]$ and $\lambda > 0$.

This balance equation is completed by boundary conditions to get the system:

$$\left\{ \begin{array}{rcll} -\operatorname{div}\sigma_s(u) & = & f & in \quad \Omega, \\ u & = & 0 & on \quad \Gamma_D, \\ \sigma_s(u)n & = & g & on \quad \Gamma_N, \end{array} \right. \tag{2}$$

where g is the surface force density and n is the unit outward normal vector to Γ.

In the sequel, we will use the following notations: If $\sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in (L^2(\Omega))^{2\times 2}$, then we denote by $\sigma : \tau = \sum_{i,j} \sigma_{ij}\tau_{ij}, \ \ (\sigma, \tau) = \int_\Omega \sigma : \tau dx$.

For shortness the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$ and in the case $D = \Omega$, we will drop the index Ω.

Finally the notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b, of the meshsize of the triangulation and of the parameter λ (but it may depend on μ_1, μ_2 and Ω), such that $a \leq C\ b$.

The variational formulation of (2) is wellknown (see section I.1.2 of [4]), and is summarized in the next Lemma.

Lemma 2.1 *Let* $f \in (L^2(\Omega))^2$ *and* $g \in (H^{-\frac{1}{2}}(\Gamma_N))^2$*, then there exists a unique solution* $u \in (H^1_{0,\Gamma_D}(\Omega))^2$ *of*

$$\int_\Omega (2\mu\varepsilon(u) : \varepsilon(v) + \lambda tr\varepsilon(u) tr\varepsilon(v))dx = \int_\Omega f v dx + < g, v >_{\Gamma_N}, \forall v \in (H^1_{0,\Gamma_D}(\Omega))^2. \quad (3)$$

For the mixed formulation of problem (3), we introduce the additional unknowns

$$\sigma = 2\mu\varepsilon(u), \quad p = -\lambda \mathrm{div} u, \quad \omega = \frac{1}{2}\,\mathrm{curl}\, u, \quad \xi = -u_{|\Gamma_N}.$$

This last unknown is a Lagrange multiplier, which is introduced in order to impose the boundary condition on Γ_N (see below).

Let us further define the spaces

$$\Sigma = \{(\tau, q) \in (L^2(\Omega))^{2\times 2} \times L^2(\Omega) : \mathrm{div}(\tau - q\delta) \in (L^2(\Omega))^2\},$$
$$Q = (L^2(\Omega))^2 \times L^2(\Omega), \quad M = Q \times (H^{\frac{1}{2}}_{00}(\Gamma_N))^2.$$

For shortness we often write the pairs $(\sigma, p), (\tau, q) \in \Sigma$ by $\underline{\sigma} = (\sigma, p), \underline{\tau} = (\tau, q)$ and similarly the pairs $(u, \omega), (v, \theta) \in Q$ by $\underline{u} = (u, \omega), \underline{v} = (v, \theta)$.

With these notations the mixed variational formulation of problem (3) is: Find $(\underline{\sigma}, (\underline{u}, \xi)) \in \Sigma \times M$ such that

$$\left\{ \begin{array}{rcll} A(\underline{\sigma}, \underline{\tau}) + B(\underline{\tau}, (\underline{u}, \xi)) & = & 0 & \forall \underline{\tau} \in \Sigma, \\ B(\underline{\sigma}, (\underline{v}, \alpha)) & = & F(v, \alpha) & \forall (\underline{v}, \alpha) \in M, \end{array} \right. \quad (4)$$

where the bilinear forms $A : \Sigma \times \Sigma \to \mathbb{R}$, $B : \Sigma \times M \to \mathbb{R}$ and the linear form $F : M \to \mathbb{R}$ are defined by

$$\begin{array}{rcl} A(\underline{\sigma}, \underline{\tau}) & = & \frac{1}{2\mu}(\sigma, \tau) + \frac{1}{\lambda}(p, q), \\ B(\underline{\tau}, (\underline{v}, \alpha)) & = & (\mathrm{div}(\tau - q\delta), v) + (as(\tau), \theta) + < (\tau - q\delta)n, \alpha >_{\Gamma_N}, \\ F(\underline{v}, \alpha) & = & -\int_\Omega f v dx + < g, \alpha >_{\Gamma_N}. \end{array}$$

First, we show the equivalence between the standard and mixed formulations by the following proposition:

Proposition 2.2 $u \in (H^1_{0,\Gamma_D}(\Omega))^2$ *is solution of (3) if and only if* $((\sigma, p), ((u, \theta), \xi)) \in \Sigma \times M$ *is solution of (4), where* $\sigma = 2\mu\varepsilon(u), p = -\lambda \mathrm{div} u, \omega = \frac{1}{2}\,\mathrm{curl}\, u, \xi = -u_{|\Gamma_N}$.

The previous Proposition guarantees in particular the well posedness of problem (4).

Theorem 2.3 *There exists a unique solution* $(\underline{\sigma}, (\underline{u}, \xi)) \in \Sigma \times M$ *of the mixed variational formulation (4) such that*

$$||(\underline{\sigma}, (\underline{u}, \xi))||_{\Sigma \times M} \lesssim (1 + \frac{1}{\lambda})^2 (||f|| + ||g||_{(H^{-\frac{1}{2}}(\Gamma_N))^2}).$$

To prove this theorem, we show the inf-sup condition of B and the uniform coerciveness of A on the kernel of B and we apply Theorem I.4.1 of [3]

3 The discrete problem

Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ made of triangles K of diameter h_K, with $h = \max\{h_K, K \in \mathcal{T}_h\}$ and such that the points of $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ are vertices of $\mathcal{T}_h$.

For $K \in \mathcal{T}_h$, let us denote by b_K, the standard bubble function defined by $b_K(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x)$ where $\lambda_i, i = 1, 2, 3$, are the barycentric coordinates on K associated with the vertices of K. The set of the edges of K will be denoted by $\mathcal{E}_K$. Let now set

$$\begin{aligned} \Sigma_h &= \{(\tau_h, q_h) \in \Sigma : q_{h|K} \in \mathbb{P}_1(K) \text{ and} \\ &\qquad (\tau_h - q_h\delta)_{|K} \in (\mathbb{P}_1(K))^{2\times 2} \oplus (\mathbb{R}\,\mathrm{curl}\, b_K)^2, \forall K \in \mathcal{T}_h\}, \\ L_h^2 &= \{v_h \in (L^2(\Omega))^2 : v_{h|K} \in (\mathbb{P}_0(K))^2, \forall K \in \mathcal{T}_h\}, \\ Q_h &= \{\theta_h \in L^2(\Omega) : \theta_{h|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Here by $(\tau_h - q_h\delta)_{|K} \in (\mathbb{P}_1(K))^{2\times 2} \oplus (\mathbb{R}\,\mathrm{curl}\, b_K)^2$ we mean that there exist polynomials p_{11}, p_{12}, p_{21}, p_{22} of degree ≤ 1 and two real numbers a_1 and a_2 such that

$$\tau_h - q_h\delta_{|K} = \begin{pmatrix} p_{11} + a_1\frac{\partial b_K}{\partial x_2} & p_{12} - a_1\frac{\partial b_K}{\partial x_1} \\ p_{21} + a_2\frac{\partial b_K}{\partial x_2} & p_{22} - a_2\frac{\partial b_K}{\partial x_1} \end{pmatrix}.$$

Let $\{I_1, ..., I_m\}$ be the partition of Γ_N induced by the triangulation $\mathcal{T}_h$, i.e., each $I_i = K \cap \bar{\Gamma}_N$ for some triangle K of $\mathcal{T}_h$ and $\bar{\Gamma}_N = \cup_{j=1}^m I_j$. Due to our previous hypotheses on the triangulation $\mathcal{T}_h$, each I_i is contained in one side of the polygonal line Γ.

Let us finally set $H_h^{\frac{1}{2}} = \{\alpha_h \in H_{00}^{\frac{1}{2}}(\Gamma_N) : \alpha_{h|I_j} \in \mathbb{P}_1(I_j), j = 1, ..., m\}$.

The approximation space of M is then defined by $M_h = L_h^2 \times Q_h \times (H_h^{\frac{1}{2}})^2$.

Contrary to [2], the space M_h is equipped with the L^2-norm, namely

$$\|((v_h, \theta_h), \alpha_h)\|_{\tilde{M}} := \|v_h\| + \|\theta_h\| + \|\alpha_h\|_{\Gamma_N}.$$

The main reason is that we want to use non quasi-uniform meshes for which the uniform inf-sup condition with the term $\|\alpha_h\|_{(H^{1/2}(\Gamma_N))^2}$ instead of $\|\alpha_h\|_{\Gamma_N}$ seems to be difficult to prove.

Accordingly the discrete problem associated with the (continuous) mixed problem (4) is: Find $\underline{\sigma}_h = (\sigma_h, p_h) \in \Sigma_h$, and $(\underline{u}_h = (u_h, \omega_h), \xi_h) \in M_h$ such that

$$\left\{ \begin{array}{llcll} A(\underline{\sigma}_h, \underline{\tau}_h) + & B(\underline{\tau}_h, (\underline{u}_h, \xi_h)) & = & 0 & \forall \underline{\tau}_h \in \Sigma_h, \\ & B(\underline{\sigma}_h, (\underline{v}_h, \alpha_h)) & = & F(\underline{v}_h, \alpha_h) & \forall (\underline{v}_h, \alpha_h) \in M_h. \end{array} \right. \tag{5}$$

To get appropriated error estimates, we need to show that the discrete inf-sup condition holds, as well as uniform coerciveness on the discrete kernel of B. For these purposes, we use the BDM_1 interpolation operator defined in [3, 14]. This allow us to show the

Theorem 3.1 *There exists $\beta_3 > 0$ independent of h such that*

$$\sup_{\underline{\tau}_h \in \Sigma_h, \underline{\tau}_h \neq 0} \frac{B(\underline{\tau}_h, (\underline{v}_h, \alpha_h))}{||\underline{\tau}_h||_\Sigma} \geq \beta_3 ||(\underline{v}_h, \alpha_h)||_{\tilde{M}}, \forall (\underline{v}_h, \alpha_h) \in M_h.$$

Lemma 3.2 *The bilinear form A is uniformly coercive with respect to λ on*

$$V_h = \{\underline{\tau}_h \in \Sigma_h : B(\underline{\tau}_h, (\underline{v}_h, \alpha_h)) = 0, \forall (\underline{v}_h, \alpha_h) \in M_h\},$$

in other words $A(\underline{\tau}_h, \underline{\tau}_h) \gtrsim \|\tau_h\| + \|q_h\|, \forall \underline{\tau}_h = (\tau_h, q_h) \in V_h$.

This Lemma and Theorem 3.1 guarantee the existence and uniqueness of a solution to problem (5).

4 Some regularity results

Let us decompose $\Gamma = \cup_{j=1}^{n_e} \overline{\Gamma}_j$, where each Γ_j is an open segment. Denote furthermore by S_j the common vertex between Γ_j and Γ_{j+1} (modulo n_e) and by ω_j the interior opening of Ω at S_j. We will distinguish three kinds of vertices, namely the set $\mathcal{S}_{DD}$ of Dirichlet-Dirichlet vertices, in the sense that S_j belongs to $\mathcal{S}_{DD}$ if and only if Γ_j and Γ_{j+1} are included into Γ_D; similarly S_j belongs to the Neumann-Neumann set $\mathcal{S}_{NN}$ if and only if Γ_j and Γ_{j+1} are included into Γ_N; and finally S_j belongs to the Dirichlet-Neumann set $\mathcal{S}_{DN}$ if and only if either Γ_j is included in Γ_D and Γ_{j+1} is included into Γ_N, or the converse. Later on, we will denote by (r_j, θ_j) the polar coordinates centered at the vertex S_j.

It is wellknown (see [9] or [8, 5]) that the weak solution of problem (2) presents vertex singularities. To describe them, we need to introduce the following notations: to each vertex S_j, we associate the following characteristic equation:

$$\begin{cases} \sin^2(\alpha\omega_j) = (\frac{\lambda+\mu}{\lambda+3\mu})^2 \alpha^2 \sin^2 \omega_j & \text{if} \quad S_j \in \mathcal{S}_{DD}, \\ \sin^2(\alpha\omega_j) = \alpha^2 \sin^2 \omega_j & \text{if} \quad S_j \in \mathcal{S}_{NN}, \\ \sin^2(\alpha\omega_j) = \frac{(\lambda+2\mu)^2 - (\lambda+\mu)^2 \alpha^2 \sin^2 \omega_j}{(\lambda+\mu)(\lambda+3\mu)} & \text{if} \quad S_j \in \mathcal{S}_{DN}. \end{cases} \tag{6}$$

Denote by Λ_j the set of complex roots of this equation. We denote by $\nu(\alpha)$ the multiplicity of $\alpha \in \Lambda_j$, it is wellknown that it is either 1 or 2.

The next result was shown in [9]:

Theorem 4.1 *Assume that characteristic equation (6) has no root on the vertical line $\Re\alpha = 1$ and that $f \in (L^2(\Omega))^2$. Then the weak solution u of problem (2) admits the following decomposition*

$$u = u_R + \sum_{j=1}^{n_e} \sum_{\alpha \in \Lambda_j : \Re\alpha \in]0,1[} r_j^\alpha \sum_{k=0}^{\nu(\alpha)-1} c_{j,\alpha,k} (\ln r_j)^k \varphi_{j,\alpha,k}(\theta_j), \tag{7}$$

where u_R belongs to $(H^2(\Omega))^2$ is the regular part of u, $c_{j,\alpha,k} \in \mathbb{C}$ is a so-called coefficient of singularity and $\varphi_{j,\alpha,k}$ is a smooth function (explicitely known, cf. [9]).

The above decomposition allows to show that u belongs to appropriated weighted Sobolev spaces that we next define.

Definition 4.2 *For any scalar function $\phi \in C^0(\overline{\Omega})$ such that $\phi(x) > 0$ $\forall x \in \overline{\Omega} \setminus \{S_1, ..., S_{n_e}\}$, and any $m, k \in \mathbb{N}$, we define*

$$H_\phi^{m,k}(\Omega) = \{v \in H^m(\Omega) : \phi D^\beta v \in L^2(\Omega), \forall \beta \in \mathbb{N}^2 : m < |\beta| \leq m + k\}.$$

$H_\phi^{m,k}(\Omega)$ is a Hilbert space with the norm $||v||_{m,k;\phi,\Omega} = (||v||_{m,\Omega}^2 + \sum_{m<|\beta|\leq m+k} ||\phi D^\beta v||^2)^{\frac{1}{2}}$
We also define the semi-norm: $|v|_{m,k;\phi,\Omega} = (\sum_{|\beta|=m+k} ||\phi D^\beta v||^2)^{\frac{1}{2}}$.

For all $j \in \{1, 2, ..., n_e\}$, we now fix a non negative real number $\alpha_j < 1$ such that

$$\alpha_j > 1 - \Re\alpha, \forall \alpha \in \Lambda_j : \Re\alpha \in]0, 1[.$$

Corollary 4.3 *Let the assumptions of Thereom 4.1 be satisfied. Let us fix* $\phi \in C^0(\overline{\Omega})$ *be as in Definition 4.2 and such that* $\phi = r_j^{\alpha_j}$ *in a neighbourhood of the vertex* S_j *for every* $j = 1, 2, \cdots, n_e$. *Then* $u \in (H_\phi^{1,1}(\Omega))^2$ *and consequently* $\sigma = 2\mu\varepsilon(\mu) \in (H_\phi^{0,1}(\Omega))^{2\times 2}, p = -\lambda \mathrm{div} u \in H_\phi^{0,1}(\Omega)$ *and* $\omega = \frac{1}{2}\,\mathrm{curl}\, u \in H_\phi^{0,1}(\Omega)$.

For further purposes, we need to give a meaning to the traces of functions in $H_\phi^{0,1}(\Omega)$, namely we show the

Lemma 4.4 *Let* ϕ *be a function like in Corollary 4.3. If* $w \in H_\phi^{0,1}(\Omega)$, *then for all triangles* $K \in \mathcal{T}_h$, *it holds* $w_{|E} \in L^1(E), \forall E \in \mathcal{E}_K$.

5 Error estimates

In this section, we take advantage of the previous results and some interpolation error estimates to obtain convergence results. We first introduce a kind of Fortin operator ([7]):

Proposition 5.1 *Let* ϕ *be a function like in Corollary 4.3. Then there exists an operator*

$$\begin{array}{rlcl} \Pi_h : & \Sigma \cap (H_\phi^{0,1}(\Omega))^{2\times 2} \times H_\phi^{0,1}(\Omega) & \longrightarrow & \Sigma_h \\ & \underline{\tau} = (\tau, q) & \longrightarrow & \Pi_h \tau = (\tau_h, q_h) \end{array}$$

$$\text{such that } B(\underline{\tau} - \Pi_h \underline{\tau}, (\underline{v}_h, \alpha_h))) = 0, \qquad \forall (\underline{v}_h, \alpha_h) \in M_h. \tag{8}$$

Corollary 5.2 *Under the assumptions of the previous proposition, we have*

$$||\underline{\tau} - \Pi_h \underline{\tau}|| \lesssim ||(\tau - q\delta) - (\tau_h^* - q_h\delta)|| + ||q - q_h||. \tag{9}$$

We now need to define local weighted Sobolev spaces:

Definition 5.3 *Let* K *be an arbitrary triangle in the plane and a vertex* A *of* K. *For* $m = 0$ *or* 1 *and* $\beta \in [0, 1[$, *we will denote*

$$H_A^{m,1;\beta}(K) = \{\psi \in H^m(K); |x - A|^\beta D^\alpha \psi \in L^2(K) \forall \alpha \in \mathbb{N}^2 : |\alpha| = m + 1\},$$

equipped with the norm $||\psi||_{m,1;\beta,K} = (||\psi||_{m,K}^2 + |\psi|_{m,1;\beta,K}^2)^{\frac{1}{2}}$ *and semi-norm* $|\psi|_{m,1;\beta,K} = (\sum_{|\alpha|=m+1} |||x - A|^\beta D^\alpha \psi||_K^2)^{\frac{1}{2}}$.

By Lemma 4.4, the trace of an element of $H_A^{0,1;\beta}(K)$ with $\beta \in [0, 1[$ is well defined and is in $L^1(\partial K)$. Thus given $v \in [H_A^{0,1;\beta}(K)]^2$, its Brezzi-Douglas-Marini interpolant $\rho_K v \in BDM_1(K) = (\mathbb{P}_1(K))^2$ [3, p.125] is well defined by the relations:

$$\int_{\partial K} \rho_K v \cdot n p_1 ds = \int_{\partial K} v \cdot n p_1\, ds, \quad \forall p_1 \in \mathcal{R}_1(\partial K).$$

Using the so-called Piola transformation and Bramble-Hilbert arguments, Farhloul and Paquet have shown in Proposition 4.12 from [7] the next result:

Lemma 5.4 *Let* $(\mathcal{T}_h)_{h>0}$ *be a regular family of triangulations of* $\overline{\Omega}$. *For any* $\beta \in [0, 1[$, *and every* $K \in \mathcal{T}_h$, *it holds* $||v - \rho_K v||_K \lesssim h_K^{1-\beta} |v|_{0,1;\beta,K}, \forall v \in (H_A^{0,1;\beta}(K))^2$.

Direct consequences of this Lemma are the next global interpolation error estimates under appropriate refinement conditions on the regular family of triangulations $(\mathcal{T}_h)_{h>0}$ (see Theorem 4.13 and its Corollary in [7]):

Theorem 5.5 *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies the two following refinement rules:*

1. If K is a triangle of $\mathcal{T}_h$ admitting S_j as a vertex, then $$h_K \lesssim h^{\frac{1}{1-\alpha_j}}, \qquad (10)$$
where α_j has been defined in section 4.

2. If K is a triangle of $\mathcal{T}_h$ admitting no S_j as a vertex, then $$h_K \lesssim h \inf_{x\in K} \phi(x), \qquad (11)$$

where ϕ is a function like in Corollary (4.3).

Then for every vector field $v \in (H^{0,1}_\phi(\Omega))^2$, it holds $$||v-\rho_h v|| \lesssim h|v|_{0,1;\phi,\Omega}, \qquad (12)$$
where $\rho_h v$ denotes the BDM_1 interpolant of v, i.e., for all $K \in \mathcal{T}_h$, $(\rho_h v)_{|K} = \rho_K v$.

Similarly for every $q \in H^{0,1}_\phi(\Omega)$, it holds: $$||q - P^1_h q|| \lesssim h|q|_{0,1;\phi,\Omega} \qquad (13)$$

where we recall that P^1_h denotes the L^2-orthogonal projection on Q_h.

Corollary 5.6 *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ satisfying the refinement conditions (10) and (11). Then for every $\underline{\tau} = (\tau, q) \in (H^{0,1}_\phi(\Omega))^{2\times 2} \times H^{0,1}_\phi(\Omega)$*
$$||\underline{\tau} - \Pi_h\underline{\tau}|| \lesssim h(|\tau|_{0,1;\phi,\Omega} + |q|_{0,1;\phi,\Omega}). \qquad (14)$$

Lemma 5.7 *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ satisfying the refinement conditions (10) and (11). For $v \in H^{1,1}_\phi(\Omega) \cap H^1_{0,\Gamma_D}(\Omega)$, denote by $L_h v$ its $\mathbb{P}_1$-Lagrange interpolant in $H^{\frac{1}{2}}_h$, in the sense that $L_h v$ is the unique element in $H^{\frac{1}{2}}_h$ such that $L_h v(x) = v(x)$, for all nodal points $x \in \bar{\Gamma}_N$ (which is meaningful). Then for all triangle $K \in \mathcal{T}_h$ having*

an edge E included into $\bar{\Gamma}_N$, it holds $$\|v - L_h v\|_E \lesssim h^{1/2}_K h|v|_{1,1;\phi,K}. \qquad (15)$$

In particular, we clearly have $$||v - L_h v||_{\Gamma_N} \lesssim h|v|_{1,1;\phi,\Omega}. \qquad (16)$$

Using the previous interpolation error estimate, we can prove the next error estimate:

Theorem 5.8 *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ satisfying the refinement conditions (10) and (11). Let $(\sigma, p), ((u, \omega), \xi)$ be the unique solution of problem (4) and let $(\sigma_h, p_h), ((u_h, \omega_h), \xi_h)$ be the unique solution of problem (5). We suppose that $f \in (L^2(\Omega))^2$ and that the characteristic equation (6) (cf. Theorem 4.1) has no root on the vertical line $\Re(\alpha) = 1$ for each $j = 1, 2, ..., n_e$. Then the next error estimate holds*

$$||\underline{\sigma} - \underline{\sigma}_h|| \lesssim (1 + \frac{1}{\lambda})h(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}), \qquad (17)$$
$$||u - u_h|| + ||\omega - \omega_h|| + ||\xi - \xi_h||_{\Gamma_N} \lesssim (1 + \frac{1}{\lambda})^2 h(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}). \qquad (18)$$

Proof: First we show that $||\pi_h - \underline{\sigma}_h|| \lesssim h(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega})$. Whence, (17) follows from this last estimate , (14) and triangle inequality. Next, we show that

$$||P_h^0 u - u_h|| + ||P_h^1 w - w_h|| + ||L_h \xi - \xi_h||_{\Gamma_N} \lesssim (1 + \frac{1}{\lambda})^2 h(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}) \quad (19)$$

Then, by standard scaling arguments, it holds $||u - P_h^0 u|| \lesssim h|u|_{1,\Omega}$,
where P_h^0 is the standard L^2-orthogonal projection of L_h^2.
Therefore, (18) follows from this last estimate, (14), (16) and (19). ■

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