Numerical simulation of exhaust muffler

An homogenized finite element method

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RÉSUMÉ. L'objectif de ce travail est de calculer l'atténuation acoustique dans un pot d'échappement contenant des tubes perforés. On se place en régime harmonique et en absence d'écoulement, ceci nous amène à résoudre l'équation de Helmholtz. L'utilisation de techniques d'homogénéisation, nous permettent de prendre en compte la présence des tubes perforés à l'aide de condition d'impédance équivalente. Pour employer la méthode des éléments finis, nous utilisons des conditions aux limites transparentes de type condition de rayonnement à distance fini dans les tubes d'entré et de sortie.

ABSTRACT. This study is interested in the acoustic diffraction by a muffler containing perforated ducts in the three-dimensional case. In harmonic time regime and for stationary fluid, this leads to solve the Helmholtz equation in three dimensions. A method which combines the multi-scales method of homogenization theory and the matched assymptotic expansion method is used and we derive a homogenized impedance condition on the perforated boundary. To use finite element method, we bounded the domaine by two fictitous boundaries where we set transparent boundary condition called radiation condition at finite distance.

MOTS-CLÉS : Silencieux d'échappement, équation de Helmholtz, tube perforé, homogénéisation, méthode des éléments finis, conditions aux limites transparentes.

KEYWORDS : Exhaust muffler, Helmholtz equation, perforated duct, homogenization, finite element method, transparent boundary condition.

1. Introduction

The use of mufflers for exhaust noise attenuation in vehicles and machinery has pushed the researchers to develop the numerical modeling methods. Some assumptions are generally used to obtain a one-dimensional approximation, it's the case of the Transfer Matrix Method (TMM) (cf. Munjal et al[6], Gerges et al[4]). Consequently, three-dimensional effects are neglected . However some studies use the finite element method for the numerical computation (cf. Wang[11], Ross[8], Tanaka et al[10]). Both the size of the holes and the period of perforations are small compared to the dimensions of the muffler, it's why using the Finite Element method is costly and the more difficult point is to generate a mesh. Consequently, a method that combines the multi-scales method of homogenization theory [2], [9] and the matched asymptotic expansion method is used, and a homogenized impedance condition on the perforated boundary is derived. In this article, we extend the two-dimensional investigation carried out in [2] in the three dimensional case. We compare our numerical results with those obtained by means of an experimental setup.

The muffler is represented by a cylindrical or elliptical box crossed by a tube. Let L be the length of the box. To simplify the presentation, the duct is assumed to be a circular cylinder, with radius R. The tube is periodically perforated inside the box. In order to use a finite element method, the domain is bounded by two fictitious boundaries located in the inlet and the outlet of the duct. Finally, the muffler is represented by Ω_ε, an open bounded domain of $I\!R^3$, as shown in figure 1. The perforated wall Σ_ε is located at $r = R$. ε denotes the center-to-center distance between every any two holes. $\varepsilon_L = \dfrac{\varepsilon}{L}$ is supposed to be a small parameter without dimensions. In the following sections, let $L = 1$. So $\varepsilon = \varepsilon_L$. Let $\gamma_{E,W}$ be two positive real numbers. The fictitious boundaries $\Gamma_{E,W}$ are defined in the cylindrical coordinates (r, θ, z) by $z = \gamma_{E,W}$, $0 < r < R$ and $0 < \theta < 2\pi$.

Let Ω be the domain composed by the muffler without the perforated wall as shown on figure 1. Then $\Omega_\varepsilon = \Omega \setminus \Sigma_\varepsilon$. Let : $\Sigma = \{(R, \theta, z);\ 0 < \theta < 2\pi,\ \ 0 < z < L\}$, $\Omega^- = \{x \in \Omega \text{ with } r > R\}$ and $\Omega^+ = \{x \in \Omega \text{ with } r < R\}$.

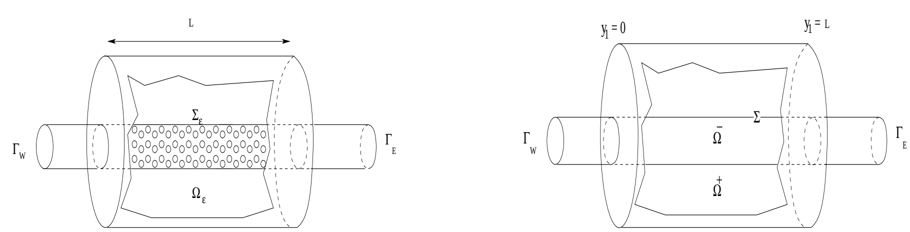

Figure 1. *The domain Ω_ε* *The domain Ω .*

An important parameter for the design of mufflers is the porosity $\sigma \in [0,1]$. It's given by : $\sigma = Lim_{\varepsilon \longrightarrow 0} \dfrac{|\Sigma| - |\Sigma_\varepsilon|}{|\Sigma|}$.

We use in the two fictitious boudaries $\Gamma_{E,W}$, a transparent condition derived from a simplified radiation condition. Finally, the total field u_ε solves the following problem :

$$\left\{ \begin{array}{lll} \Delta u_\varepsilon + k^2 u_\varepsilon & = 0 & \text{in} \quad \Omega_\varepsilon \\ \dfrac{\partial u_\varepsilon}{\partial n} & = 0 & \text{on} \quad \partial\Omega_\varepsilon \setminus \Gamma_{E,W} \\ \dfrac{\partial u_\varepsilon}{\partial n} - iku_\varepsilon & = f_{E,W} & \text{on} \quad \Gamma_{E,W} \end{array} \right. \qquad [1]$$

where $f_{E,W}$ derives from the incident field.

2. Homogenization method

This work is an extension for the three-dimensional case, of the method introduced in [2]. The solution of problem (1) is assumed to be expressed in the domain $\Omega^+ \cup \Omega^-$, as an *outer expansion* :

$$u_\varepsilon = u_0^\pm(x) + \varepsilon u_1^\pm(x) + ... \ \text{ in } \ \Omega^\pm \qquad [2]$$

Where u_j^+ (the restriction of u_j to Ω^+) and u_j^- (the restriction of u_j to Ω^-) are two independent functions. Substituting the solution by its expansion (2) in problem (1), allows to determine the problems satisfied by $u_j^\pm$ for every $j \geq 0$. The relationship between these two functions will be established later on after matching each of them with a new expansion, available in the vicinity of the perforated wall Σ_ε, called the *inner expansion*.

The effects caused by the extremities of the perforated duct are neglected and a new system of coordinates (O, y_1, y_2, y_3) is introduced. The image of Σ_ε by the mapping $(r, \theta, z) \longrightarrow y = (y_1, y_2, y_3) = (\dfrac{r-R}{\varepsilon}, \dfrac{R\theta}{\varepsilon}, \dfrac{z}{\varepsilon})$ becomes, as ε tends to zero, a boundary Σ_1 which is infinite in all directions, and periodic with respect to y_2 (resp. y_3), with period b (resp. 1). This boundary is composed by small cells obtained from the reference cell $\Gamma = \{y = (y_1, y_2, y_3) \in I\!R_y^3, / y_1 = 0, y_2^2 + y_3^2 < \sqrt{(1 - \sigma\frac{b}{\pi}}\}$ by periodicity.

In the following, the domain $G = (I\!R \times]0, b[\times]0,1[) \setminus \Gamma$ (see figure 2) is considered. G is a period of the domain study in the new frame.

The function u_ε is assumed to expand in the vicinity of the perforated wall Σ_ε as follows :

$$u_\varepsilon = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ... + \varepsilon^m v_m + ... \qquad [3]$$

where each term v_j does not depend on ε, and satisfies :

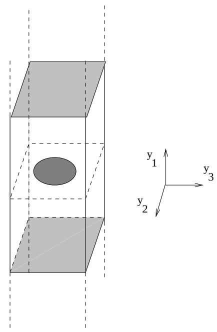

Figure 2. *The cell G.*

$$v_j = v_j(\theta, z, y)\ ;\ y = (\frac{r-R}{\varepsilon}, \frac{R\theta}{\varepsilon}, \frac{z}{\varepsilon})$$
$$v_j \text{ is } b \ - \text{periodic w.r.t } y_2 \text{ and } \ 1 \ - \text{periodic w.r.t } \ y_3)$$

The local variables y_2 and y_3 are related to the periodicity effects, while y_1 takes into account the boundary layer effects.

To ensure the coherence and the unicity of inner and outer expansions some matching rules have to be established, by refering to [2], the following results are obtained :

$u_0 \in H^1(\Omega)$ and solves the following problem :

$$\begin{cases} \Delta u_0 + k^2 u_0 & = & 0 & \text{in } \Omega \\ \dfrac{\partial u_0}{\partial n} & = & 0 & \text{on } \partial\Omega \setminus \Gamma_{E,W} \\ \dfrac{\partial u_0}{\partial n} - iku_0 & = & f_{E,W} & \text{on } \Gamma_{E,W} \end{cases} \qquad [4]$$

and the first order term u_1 satisfies :

$$\begin{cases} \Delta u_1 + k^2 u_1 & = & 0 & \text{in } \Omega^+ \cup \Omega^- \\ \dfrac{\partial u_1}{\partial n} & = & 0 & \text{on } \partial\Omega \setminus \Gamma_{E,W} \\ [u_1] & = & \beta \frac{\partial u_0}{\partial r} & \text{on } \Sigma \\ \left[\dfrac{\partial u_1}{\partial n}\right] & = & 0 & \text{on } \Sigma \\ \dfrac{\partial u_1}{\partial n} - iku_1 & = & 0 & \text{on } \Gamma_{E,W} \end{cases} \qquad [5]$$

Where $\beta = \beta^+ - \beta^-$, $\beta^\pm$ are defined from the asymptotic behavior of the term v_1. In the two-dimensional case, the calculation of $\beta^\pm$ are carried out analytically. In the treedimensional case this coefficient has to be determined numerically, by solving a partial differential equation, introduced in the next section.

3. The first order approximation

Let $u_{\varepsilon,1} = u_0 + \varepsilon u_1$ be a first order approximation of u_ε . The function u_0 solves problem (4) and u_1 solves (5). Hence, $u_{\varepsilon,1}$ satisfies the following equation on Σ :

$$[u_{\varepsilon,1}] = \varepsilon\beta\frac{\partial u_0}{\partial r} = \varepsilon\beta\frac{\partial u_{\varepsilon,1}}{\partial r} - \varepsilon^2\beta\frac{\partial u_1}{\partial r} \text{ on } \Sigma.$$

If the second order term is neglected in the previous equation and if $\tilde{u}_{\varepsilon,1}$, is the solution of the modified equations, the following transmission problem is obtained :

$$\left\{\begin{array}{rcll} \Delta\tilde{u}_{\varepsilon,1} + k^2\tilde{u}_{\varepsilon,1} & = & 0 & \text{in } \Omega^+ \cup \Omega^- \\ \dfrac{\partial\tilde{u}_{\varepsilon,1}}{\partial n} & = & 0 & \text{on } \partial\Omega \setminus \Gamma_{E,W} \\ \left[\dfrac{\partial\tilde{u}_{\varepsilon,1}}{\partial n}\right] & = & 0 & \text{on } \Sigma \\ [\tilde{u}_{\varepsilon,1}] & = & \varepsilon\beta\dfrac{\partial\tilde{u}_{\varepsilon,1}}{\partial r}(R,\theta,z) & \text{on } \Sigma \\ \dfrac{\partial\tilde{u}_{\varepsilon,1}}{\partial n} - ik\tilde{u}_{\varepsilon,1} & = & f_{E,W} & \text{on } \Gamma_{E,W} \end{array}\right. \qquad [6]$$

The impedance coefficient $\beta = \beta^+ - \beta^-$ used in the transmission condition on the boundary Σ, takes into account the presence of the perforated wall. This coefficient is obtained (cf. [2]) from the assymptotic behaviour of the first order term v_1 of the inner expansion. Indeed, v_1 can be expressed under the following form : $v_1 = \dfrac{\partial u_0}{\partial r}(\varphi + y_1)$, where φ solves a partial differential equation in the cell G and $\varphi \sim \beta^\pm$ when $y_1 = \pm\infty$.

In order to compute φ by a finite element method, we bound the cell G by two fictitous boundaries Γ_1 and Γ_2 located respectively on $y_1 = \gamma_1$ and $y_1 = \gamma_2$. Ĝ = ( ]0, 1[$\times$]0, b[$\times$]γ_1, γ_2[ ) is the truncated domain. Let Γ_+ (resp. Γ_-) be the side of Γ in the half space $y_1 > 0$ (resp. $y_1 < 0$) and $\Gamma_r = \partial\hat{G} \setminus (\Gamma_+ \cup \Gamma_- \cup \Gamma_1 \cup \Gamma_2)$. On $\Gamma_{1,2}$, φ satisfies transparent boundary conditions derived from a modal decomposition. φ solves the following problem :

$$\left\{ \begin{array}{lcll} \triangle_y \varphi & = & 0 & \text{in } \hat{\mathrm{G}} \\ \dfrac{\partial \varphi}{\partial y_1} & = & -1 & \text{on } \Gamma_+ \\ \dfrac{\partial \varphi}{\partial y_1} & = & +1 & \text{on } \Gamma_- \\ \dfrac{\partial \varphi}{\partial n} & = & 0 & \text{on } \Gamma_r \\ \dfrac{\partial \varphi}{\partial n} & = & T_{1,2}(\varphi|_{\Gamma_{1,2}}) & \text{on} \, \Gamma_{1,2} \end{array} \right. \qquad [7]$$

Where $\triangle_y = \dfrac{\partial^2}{\partial y_1^2} + \dfrac{\partial^2}{\partial y_2^2} + \dfrac{\partial^2}{\partial y_3^2}$ and $T_{1,2}$ is the Dirichlet-to-Neumann operator mapping $H^{1/2}(\Gamma_{1,2})$ into $H^{-1/2}(\Gamma_{1,2})$.

4. Numerical results

The numerical results are obtained with the MELINA code [5]. Two models of exhaust muffler are considered, with a high porosity perforated concentric tube inside an elliptic expansion chamber. The second muffler is an extended perforated tube at the inlet and a concentric perforated plugged tube. The pressure is evaluated, by a finite element method (FEM) discretization applied on the first order approximation. The trasmission loss curves obtained by the finite element method are compared to the experimental and transfer matrix (TMM) results presented in [4]. Figure (3) and (4) shows that for low frequencies (under 1000 Hz) the results obtained by TMM method are comparable to FEM and experimental results. For higher frequencies, the FEM results are better. The peaks of the transmission loss curves are determined by the FEM, but with a small translation.

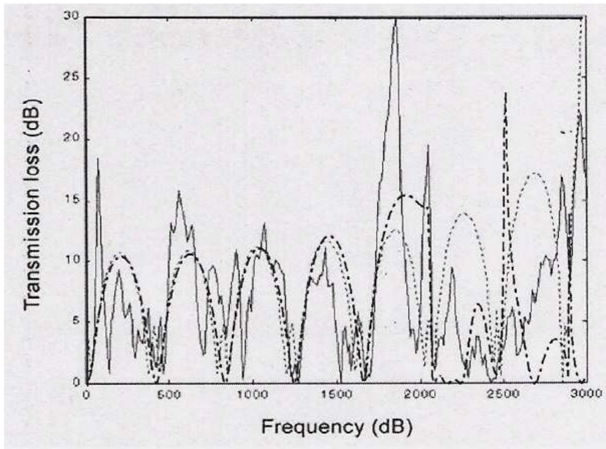

Figure 3. *Transmission loss of a concentric perforated tube ; FEM results (dot-dashed line) ,TMM results (dashed line) and experimental results (solid line).*

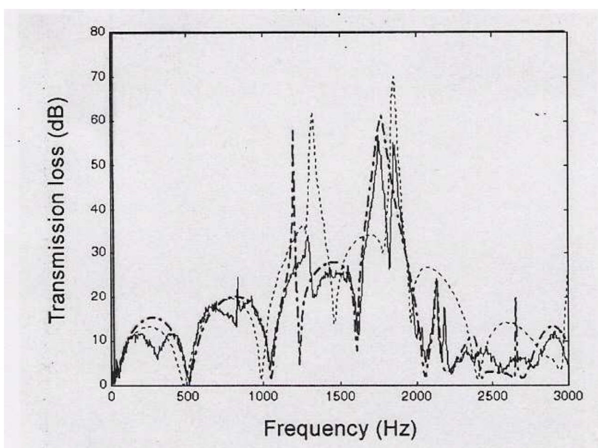

Figure 4. *Transmission loss of a concentric perforated plugged tube ; FEM results (dot-dashed line) ,TMM results (dashed line) and experimental results (solid line).*

Figure 5. *Isovalues at low frequencies of an expansion room crossed by a perforated tube.*

Figure 6. *Isovalues at higt frequencies of an expansion room crossed by a perforated tube.*

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