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On solving the data recovering problem on a flat boundary; applications to a class of inverse problems

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RÉSUMÉ. En suivant les travaux de J. Cheng *et al* [7], on reformule le problème de complétion de données en un problème de reconstruction d'une fonction connaissant ses moments. Cette interprétation se fait via la simple utilisation d'une formule de Green. Des tests numériques illustrent l'efficacité de la méthode proposée. Nous appliquons également cette méthode à la résolution de deux problèmes inverses correspondants à la détermination d'un coefficient de Robin et à l'identification d'une fissure d'interface.

ABSTRACT. Following J. Cheng *et al* [7], by using the Green's formula we rephrase the illposed problem of boundary data recovering as a moment problem. Robustness of the proposed missing data reconstruction process is investigated. Numerical tests highlight the efficiency of the proposed method. In addition we give an application to two inverse problems, Robin coefficient determination and interfacial crack recovery.

MOTS-CLÉS : Problème de Cauchy, polynôme de Legendre, problèmes des moments, coefficient de Robin, fissures

KEYWORDS : Cauchy problem form Laplace equation, Legendre polynomial, moment problem, Robin coefficient, cracks identification.

1. Introduction

We consider in this work the problem of recovering lacking data on some part of the boundary of a domain from overspecified boundary data on the remaining part of the boundary. This kind of problem may occur very often in engineering sciences as the reconstruction of physical variables from lacking data is highly useful in many industrial processes. The more common problem borrowed from thermostatic consists in recovering the temperature in a given domain when the distribution of it and of the heat flux along the accessible region of the boundary are known. Given a flux Φ and the corresponding temperature T on Γ_c, one wants to recover the corresponding flux and temperature on the remaining part of the boundary Γ_i, where Γ_c and Γ_i constitute a partition of the whole boundary $\partial\Omega$. The problem is therefore set as follows

Find (φ, t) on Γ_i such that there exists a temperature field u satisfying :

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \nabla u.n = \Phi & \text{on} \quad \Gamma_c, \\ u = T & \text{on} \quad \Gamma_c. \end{cases} \qquad [1]$$

This problem is known since Hadamard [2, 3, 6] to be illposed in the sense that the dependence of u and consequently of (φ, t) on the data (Φ, T) is known to be not continuous. We propose, in this paper, to reconstruct directly both lacking data (φ, t) .

There are several general algorithms for solving these problems. In our approach we will restrict ourselves to the case where Γ_i is a flat boundary. In this particular case, the data extension problem turns out to be quasi-explicitely solvable in so far as it is rephrased as a moments problem.

Our work is to be linked to those [7] in which the Cauchy problem has been transformed into a moment problem in the particular case where the boundary Γ_i, on which the data is lacking, is flat.

2. Data boundary recovering via Legendre moment problem

The following Cauchy problem

$$\Delta u = 0 \text{ in } \Omega, \qquad [2]$$

$$u = f \text{ on } \Gamma_c, \qquad [3]$$

$$\frac{\partial u}{\partial n} = \phi \text{ on } \Gamma_c. \qquad [4]$$

is well known to be highly ill posed since Hadamard, that is any small change in the initial data may induce large variation of the solution, [2, 3, 6]. In [7], J. Cheng develop a numerical method for an approximation of the solution of $(2) - (4)$. More precisely, by using the Green formula, the Cauchy problem is transformed into a moment problem. In fact they prove that the initial Cauchy problem (2)-(4) is equivalent to the following moment problem

$$\int_{\Gamma_i} v\beta ds = \mu_v(f,\phi), \qquad [5]$$

where β is an unknown function defined on $\partial\Omega\backslash\Gamma$, v is an harmonic function such that

$$\Delta v = 0 \text{ in } \Omega, \qquad [6]$$

$$\frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_i. \qquad [7]$$

To approach the solution of (5) they they found the moments of the solution on the basis $\{x^j\}_{j\in\mathbb{N}}$ and they use the Gram-Schmidt matrix to recover the Legendre moments, however this matrix is ill conditioned. Our contribution in this paper, is to make appear directly the components with respect to an orthonormal basis and apply these results to some inverse problems. Let v be an harmonic function solution of the equations

$$\Delta v = 0 \text{ in } \Omega, \qquad [8]$$

$$v = 0 \text{ on } \Gamma_i. \qquad [9]$$

We denote by $H_1 = \{v\,|\,v$ satisfies (8)-(9)$\}$ and $H_2 = \{v\,|\,v$ satisfies (6)-(7)$\}$. Consider $\{L_j\}_{j\in\mathbb{N}}$ a set of functions such that the following conditions are fulfilled

$$(A)\left\{\begin{array}{l}\overline{Span\{L_j\}_{j=0}^{\infty}} = L^2(\Gamma_i),\\ \{L_j\}_{j\in\mathbb{N}} \text{ is an orthogonal basis in } L^2(\Gamma_i).\end{array}\right.$$

L_j is a Hilbertian basis of L^2.

We will consider from now on, the Legendre-Fourier polynomials. Let us point out that we obtain here the Legendre moments of the function u and $\frac{\partial u}{\partial n}$ directly. Our approach avoid the orthonormalization step, which is highly ill posed. For that, let $P_0(x), P_1(x), \ldots,$ be a shifted Legendre polynomials normalized by $\int_0^1 (P_j(x))^2 dx = 1$. These polynomials are defined for all $j = 0, 1, \ldots$ by

$$P_j(x) = \sum_{k=0}^{j} C_{jk} x^k, \qquad [10]$$

where

$$C_{00} = 1,\ C_{j,0} = (2j+1)^{\frac{1}{2}},\ C_{j,k} = -C_{j,k-1}(\frac{j}{k}+1)(\frac{j+1}{k}-1)$$

$$\text{and } j = 0,1,2,3,\ldots \quad k = 1,2,3,\ldots,j.$$

For illustration, let Ω be a bounded, simply Lipschitz connected domain of $\mathbb{R}^2$ defined by

$$\Omega = \left\{(x,y) \in \mathbb{R}^2 : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1\right\},$$

with

$$\Gamma_i = \left\{(x,y) \in \mathbb{R}^2 : \ y = 0, \ 0 \leq x \leq 1\right\},$$

and Γ_c is a Lipschitz curve in $\left\{(x,y) \in \mathbb{R}^2 / \ y \geq 0\right\}$ which connects the two points $(0,0)$ and $(1,0)$ such that $\left\{(x,y) \in \mathbb{R}^2 / y = 0, \ 0 \leq x \leq 1\right\} \cup \Gamma_c = \partial\Omega$. To recover the unknown function, one has to construct a family of harmonic functions v_n such that

$$\frac{\partial v_n}{\partial y}(x,0) = P_n(x), n = 0, 1, \dots .$$

For that, it is sufficient to choose

$$v_n(x,y) = \mathrm{I\scriptstyle M}(Q_n(x+iy)), \ n = 0, 1, \dots$$

where Q_j is a primitive of P_j and Im is the imaginary part. For all $x \in \mathbb{R}$, we define the sequence $\{q_j\}_{j \in \mathbb{N}}$ such that $q_j(x) = \int_1^x P_j(t)dt$, for $Z = (x,y)$ in $\mathbb{R}^2$, then we deduce the sequence $Q_j(x,y) = Im(q_j(x+iy))$. We can prove easily that for all j in $\mathbb{N}$, Q_j is in H^1. By using Green's formula, we set

$$\int_{\Gamma_c} \left(\frac{\partial Q_j}{\partial n} u - \frac{\partial u}{\partial n} Q_j\right) ds = \int_{\Gamma_i} \left(\frac{\partial Q_j}{\partial n} u - \frac{\partial u}{\partial n} Q_j\right) ds. \qquad [11]$$

On Γ_i, we have $\frac{\partial Q_j}{\partial n}(x,0) = P_j(x,0)$ and $Q_j(x,0) = 0, \forall j \in \mathbb{N}$, therefore

$$\int_{\Gamma_c} \left(\frac{\partial Q_j}{\partial n} u - \phi Q_j\right) ds = \int_{\Gamma_i} P_j u ds. \qquad [12]$$

Which gives the Legendre moments of the solution of problem $(2)-(4)$ on Γ_i. To apply this method for the identification of Robin coefficient and interfacial crack recovery, we have also to reconstruct the normal derivative $\frac{\partial u}{\partial n}$ on Γ_i. For that reason we define the sequence $\{d_j\}_{j \in \mathbb{N}}$ such that $d_j(x) = \frac{dP_j}{dx}(x)$. For $Z = (x,y)$ in $\mathbb{R}^2$, we deduce the sequence $D_j(x,y) = Re(d_j(x+iy)), j \in \mathbb{N}$. Take $v_j = D_j$ in (11), then on Γ_i we have

$$\frac{\partial D_j}{\partial y}(x,0) = 0 \text{ and } D_j(x,0) = Re(P_j(x,0)), \forall j \in \mathbb{N}$$

The identity (11) becomes

$$\int_{\Gamma_c} \left(\frac{\partial D_j}{\partial n} u - \phi D_j\right) ds = \int_{\Gamma_i} P_j \frac{\partial u}{\partial n} ds \qquad [13]$$

which allow the recovering of Legendre moments of $\frac{\partial u}{\partial n}$.

By using the results found in this section, the Legendre moment $\mu_j,\ j \in \mathbb{N}$, for the solution u of the problem $(2)-(4)$, on Γ_i, are given through the following formula

$$\mu_j = \int_{\Gamma_c} \left(\frac{\partial Q_j}{\partial n} u - \phi Q_j \right) ds = \int_{\Gamma_i} L_j u ds,\ \forall j \in \mathbb{N}.$$

Hence the approximate value of u, on Γ_i, is a polynomial p_n of degree n such that $p_n(t) = \sum_{j=0}^{n} \mu_j L_j(t)$. The Legendre moment for the $\frac{\partial u}{\partial n}$ are given by

$$\kappa_j = \int_{\Gamma_c} \left(\frac{\partial D_j}{\partial n} u - \phi D_j \right) ds = \int_{\Gamma_i} L_j \frac{\partial u}{\partial n} ds,\ \forall j \in \mathbb{N}.$$

Hence the approximate value of $\frac{\partial u}{\partial n}$ is a polynomial q_n of degree n such that $q_n(t) = \sum_{j=0}^{n} \kappa_j L_j(t)$.

3. Numerical experiments

In this section, several numerical experiments are presented to verify the accuracy of the proposed method. First, we consider the problem of recovering of the solution u and $\frac{\partial u}{\partial n}$ on Γ_i. Second, we test our identification process to real experimental data, and to go towards a reconstruction based on true data, we are interested in the robustness of the algorithm with regard on simulate noisy data.

As an exact solution of $(2)-(4)$, take $u(x,y) = f_1(x,y) = exp(ay)cos(bx)$, where a and b are two real constants. Let Ω and Γ_c the set and a part of his boundary respectively as defined in section 2.

Exploiting the previous data matching procedure, we can compute $\mu_j = \int_{\Gamma_c} (f \frac{\partial v_j}{\partial n} - \phi v_j) ds$ and $\kappa_j = \int_{\Gamma_c} \left(\frac{\partial D_j}{\partial n} u - \phi D_j \right) ds$ respectively, $j = 0, 1, 2, \ldots$. The exact value u on Γ_i is described by the solid lines.

In Figure 1, we take $a = b = 5$ and $n = 5, 16, \ldots, 17$. Note that for $n \geq 17$, a large error will occurs between the exact and the approximate solution. In Figure 2, we study the convergence of the algorithm for oscillating functions, we take $a = b = 15, 17, 20$ and $n = 10$. For the same example, we present in Figure 3 the approximation of $\frac{\partial u}{\partial n}$ on Γ_i with $a = b = 5, 15, 15$ and n equal $5, 5$ and 10.

In [7], Cheng *et al.*, took the example $u(x,y) = f_2(x,y) = exp(10y) * sin(10x) + exp(3y)cos(3x) + 10(y^3 - 3yx^2)$ and they found a large error between exact and approximate solution when $n \geq 12$. As illustrated in figure 4, our data matching procedure compares very well with the one described in [7].

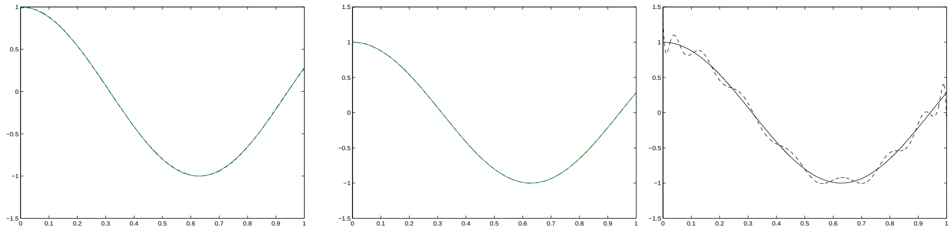

Figure 1. *Curves of $u = f_1$ and its approximation for $a = b = 5$ and for $n = 5$, $n = 16$ and $n = 17$ respectively*

4. Applications

We present here an application to two inverse problems, Robin coefficient determination and interfacial crack recovery.

4.1. The Robin inverse problem

Let u be the solution of the problem

$$\left\{ \begin{array}{l} \Delta u = 0 \text{ in } \Omega, \\ \frac{\partial u}{\partial n} = \phi \text{ on } \Gamma_c, \\ \frac{\partial u}{\partial n} + \varphi u = 0 \text{ on } \Gamma_i. \end{array} \right. \qquad [14]$$

The inverse Robin problem is to find a function φ such that the solution of the previous problem satisfies $u = f$ on M, where M is a part of accessible boundary Γ_c. In this work we suppose that $M = \Gamma_c$, our objective is to approach the function φ on Γ_i. The field, which we simulate is given by $f_2(x, y)$, see subsection 3.1. Then using the previous data matching procedure, we approach both u and $\frac{\partial u}{\partial n}$ respectively on Γ_i. We give in figure 5 the approximation of φ for $a = b = 5, 10$ and 15 and we take for the degree of the polynomial of approximation, $n = 5, 8$ and 10 respectively.

4.2. Interfacial crack recovery

In this section, we examie a numerical situation to capture a linear emerging interfacial crack with a satisfactory accuracy.

In this experience, the crack σ is known to lay in an apriori known line ω that splits the domain Ω, for which Γ_c stands for the whole external boundary, into two subdomains Ω_1 and Ω_2, each of them having a common part with Γ_c. The temperature and the heat flux on the outer boundary Γ_c are provided by (f, Φ).

The methodology for the crack detection is as follows. We consider two completion data problems set on each subdomain Ω_i,

$$\left\{ \begin{array}{l} \Delta u_i = 0 \text{ in } \Omega_i, \\ \frac{\partial u_i}{\partial n} = \phi \text{ on } \Gamma_c \cap \partial\Omega_i, \\ u_i = f \text{ on } \Gamma_c \cap \partial\Omega_i. \end{array} \right. \qquad [15]$$

Then we use the previous procedure to recover $[u] = u_1 - u_2$ on the line ω. The crack is localized as the support of $[u]$.

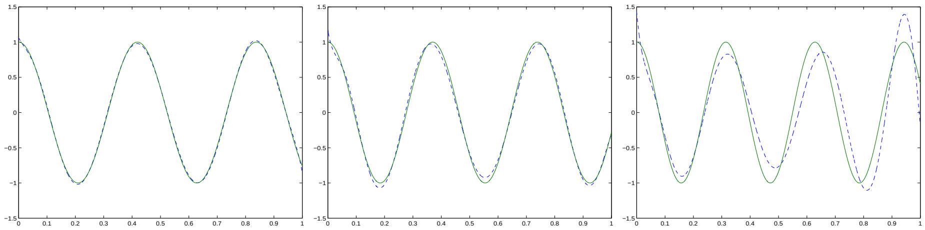

Figure 2. *Curves of $u = f_1$ and its approximation for $a = b = 15, 17, 20$ and $n = 10$*

For numerical tests we consider the function $f_3 = (r)^{1/2} cos(\frac{\theta}{2})$, the crack is on the x-axis and is specified by the part where f has a jump, it coincides with $\sigma = [0, 0.5]$. Then we can recover the solution of $(2) - (4)$ with $f = f_3$ and $\phi = \frac{\partial f_3}{\partial n}$ on Γ_c. We take for the degree of the polynomial of approximation $n = 10, 12$ and $n = 14$, see figure 6.

5. Comments

We proposed in this work a method for data matching via a Legendre moment probem. Our data recovering process has two main features that make it an efficient method. The first feature is undoubtly its robustness : It compares very well with existing data recovering processes because it allows the reconstruction of highly oscillating data. The second feature we would like to point out concerns the cost of the present method : Since we have converted the data matching probem into a moment one, the recovering process turns out to be quasi-explicit (i.e it does not require any resolution of the forward problem). We tested successfully the matching method in the case of temperature and heat flux recovering. Practical applications have been also presented.

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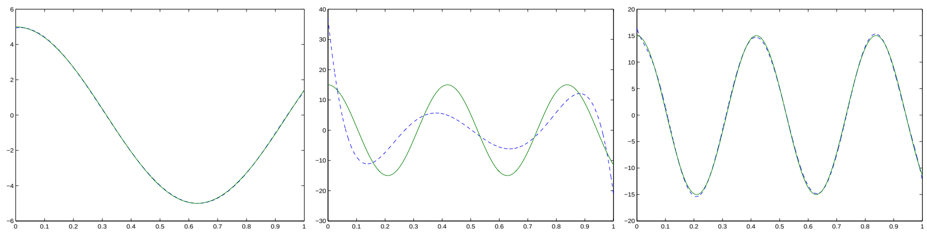

Figure 3. *Curves of* $\frac{\partial u}{\partial n}$, *for* $u = f_1$, *and its approximation for* $a = b = 5, 15, 15$ *and* $n = 5, 5, 10$

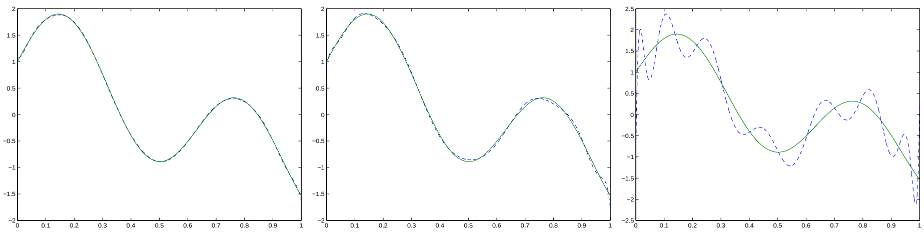

Figure 4. *Curves of* $\frac{\partial u}{\partial n}$, *for* $u = f_2$, *and its approximation for* $n = 13, 14$ *and* 15 *respectively*

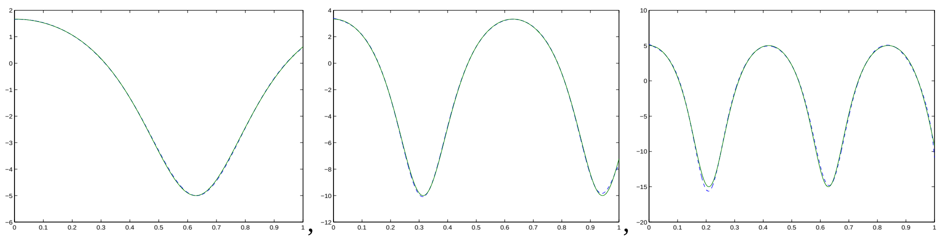

Figure 5. *Curves of* φ *with* $u = f_1$ *and its approximation for* $a = b = 5, 10$ *and* $a = b = 15$ *and* $n = 5, 8, 10$

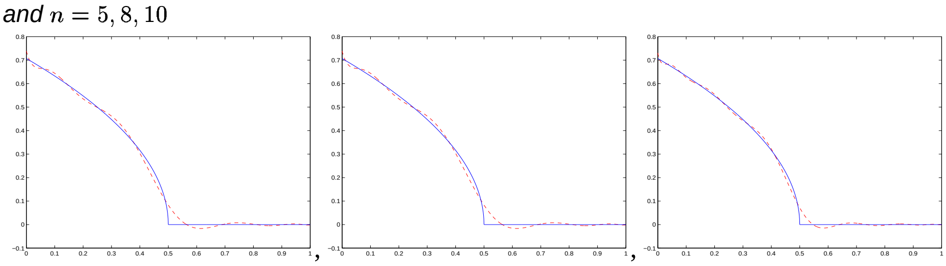

Figure 6. *Curves of the function* $u = f_3$ *and its approximation for* $n = 10, 12$ *and* 14, *on* Γ_i