Nonlinear solvers in image processing

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RÉSUMÉ. Dans ce travail, nous présentons des méthodes pour résoudre un modèle non-linéaire dans le traitement d'image. L'EDP est discrétisée par un schéma implicite de différences finies. Nous utilisons des solveurs de type Newton augmentés avec stratégies de globalisation telles que des recherches linéaires avec rebroussement. La convergence de ces méthodes peut échouer, pour éviter cet inconvénient et accélérer la convergence, des préconditionnements non-linéaires sont présentés. Un example de débruitage d'image est présenté.

ABSTRACT. In this work, we present methods to solve a nonlinear model in image denoising discretized by an implicit finite differences scheme. We use Newton like solvers augmented by globalization strategies as line search with backtracking. But convergence of these methods can fail. To avoid this and accelerate the convergence, nonlinear preconditioner are introduced. An image denoising example is presented.

MOTS-CLÉS : Méthodes de type Newton, globalisation, traitement d'images, préconditionnements non-linéaires

KEYWORDS : Newton-like methods, globalization, images processing, nonlinear preconditioners

1. Introduction

Simple models used in image denoisig are based on linear diffusion process described by the heat equation :

$$\frac{\partial u}{\partial t} - \Delta u = 0.$$

However, results of this process are not satisfactory, it not only smoothes noise but also simultaneously blurs edges. Consequently , nonlinear partial differential equations are introduced and proved to be efficient filters in image processing [3, 6, 17]. These models are generally written as follows :

$$\begin{cases} \frac{\partial u}{\partial t} - \mathrm{div}(g(|\nabla u|)\nabla u) & = & 0 & \text{in } \Omega \\ u(.,0) & = & u_0 & \text{in } \Omega \\ \frac{\partial u}{\partial n}|_{\partial\Omega} & = & 0, & \end{cases} \tag{1}$$

where $g(.)$ is the diffusivity function and controls the amount of diffusion present in the data. It is a non negative function, decreasing monotone, such that $g(0) = 1$ and $\lim_{t\to\infty} = 0$.
Perona and Malik [17] proposed the following diffusivity functions :

$$g(s) = \frac{1}{1+\frac{s^2}{\lambda^2}}, \qquad \lambda > 0$$

$$g(s) = \exp(-\frac{s^2}{\lambda^2}), \qquad \lambda > 0$$

Another example is the diffusivity function proposed by Charbonnier [7] and defined by :

$$g(s) = \frac{1}{\sqrt{1+\frac{s^2}{\lambda^2}}}, \qquad \lambda > 0.$$

In [2] and [3], the authors show the existence and uniqueness of a solution for evolutive non-linear models. In [3] the following diffusivity function is used :

$$g(s) = \frac{1}{\sqrt{1+\frac{s^2}{\lambda^2}}} + \alpha, \qquad \lambda > 0. \tag{2}$$

They prove the existence and uniqueness of a solution under some conditions on the parameter α. For numerical solutions, they use an explicit finite differences scheme in time. But it is difficult to verify the CFL condition in this case. The study of the model stability of the model depends on the parameter α. The suitable choice of α, and the step of discretization that verify the CFL condition makes the numerical solution difficult.
An adequate solution would be to use an implicit scheme, but the solution of the problem considered by an implicit scheme after linearization by the Newton method is often expensive. In addition, convergence is local and depends on the initial guess. Thus, we solve the nonlinear system by like-Newton methods for which we use globalization strategies such as line search and trust region methods. To accelerate the convergence we nonlinearly precondition the problem.
In the section below the solution of the diffusion equation (1) is introduced. We focus on

how to accelerate the convergence of Newton like methods by nonlinearly preconditioning the partial differential equation.

2. Solution of the diffusion equation

We are interested in the following to a given noised image u_0. The denoisy image is solution of the problem (1). To approximate the problem (1), an implicit finite differences scheme is used. We denote respectively by h and dt the spatial and time steps sizes. In the sequel, we take $h = 1$ and we define for every field $p = (p_1, p_2) \in \mathbb{R}^2$, the discrete divergence approximation :

$$\mathrm{div}(p)(i,j) = \begin{cases} p_1(i,j) - p_1(i-1,j) & \text{if} \quad 1 < i < N_1 \\ p_1(i,j) & \text{if} \quad i = 1 \\ -p_1(i-1,j) & \text{if} \quad i = N_1 \end{cases}$$

$$+ \begin{cases} p_2(i,j) - p_2(i,j-1) & \text{if} \quad 1 < j < N_2 \\ p_2(i,j) & \text{if} \quad j = 1 \\ -p_2(i,j-1) & \text{if} \quad j = N_2 \end{cases}$$

The discrete problem is written as :

$$u^{k+1}(i,j) - u^k(i,j) - dt\ \left(\mathrm{div}(g(|\nabla u|))^{k+1}\right)(i,j) = 0,\ \ 1 \le k \le M,$$

where $u^k(i,j) = u(x_i, y_j, t_k)$, $x_i = ih$, $y_j = jh$, $t_k = k\ dt$ and $dt = \frac{Tmax}{M}$.

In this case, we solve for each time step a nonlinear system

$$F(u) = 0,\ \ F : \mathbb{R}^N \to \mathbb{R}^N,\ \ (N = N_1 \times N_2). \tag{3}$$

using Newton like solvers that we introduce briefly in the following section.

2.1. Newton solvers

The system (3) is classically solved by Newton like methods, namely inexact Newton methods and Quasi-Newton methods. Inexact Newton methods solve approximately the Newton linear system with a suitable precision that corrects the nonlinear iteration, see [8, 11]. The convergence is then controlled of the so called forcing terms [12]. Quasi-Newton methods [9, 14, 15], on the other hand, approximate and update the Jacobian in order to avoid the costly calculation of derivatives. The most popular quasi-Newton method is Broyden method [5]. Inexact Newton and quasi-Newton methods both require more storage as iterations progress. The cost in function evaluation for quasi-Newton methods is less expensive than for inexact Newton methods. For example, Broyden method requires one function evaluation per nonlinear iteration. An auto-adaptative limited memory Broyden method is also introduced [1]. But these methods converge only locally and should be augmented by globalization strategies as line search [11] and trust regions methods [16]. To accelerate the convergence and avoid failure of Newton like methods , the iteration should be nonlinearly preconditioned where nonlinearities in partial differential equations become more balanced. The following section is devoted to the introduction of this aspect.

2.2. Nonlinear preconditioners

Even with globalization techniques, nonlinear solvers to solve (3) may stagnate to a local minima of $f \equiv \|F\|$, in particular for problems having unbalanced nonlinearities. An iteration of a Newton like method with Backtracking is written as :

$$x_{k+1} = x_k - \lambda_k s_k,$$

where λ_k is a step length computed by a line search. Let J_k the Jacobian of the function F, the search direction is often computed by solving the preconditioned linear system :

$$M_k^{-1} J_k s_k = -M_k^{-1} F_k.$$

According to [19], when the inexact Newton method fails to converge, then :

$$\frac{1}{\mathrm{cond}(J_k)} \leq \cos(\theta_k) \leq \frac{2}{\mathrm{cond}(J_k)}, \tag{4}$$

where

$$\cos(\theta_k) = \frac{-s_k^T \nabla f_k}{\| s_k \| \| \nabla f_k \|}.$$

θ_k is the angle between the search direction s_k and the negative gradient direction of $\|F\|$. Hence, when the Jacobian is ill-conditioned, the Newton direction is almost orthogonal with the gradient of $\|F\|$. In this case, the function F should be nonlinearly preconditioned.

The idea of nonlinear preconditioning is to transform the system (3) into a new nonlinear system

$$\mathcal{F}(u) = G(F(u)) = 0, \tag{5}$$

which has the same solution as the original system and where the nonlinearities are more balanced. The preconditioner $G : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is easy to compute and verify the following properties :

- If $G(x) = 0$, then $x = 0$.
- $G \simeq F^{-1}$ in some sense.

The definition of a nonlinear preconditioner can not be given precisely, nor it is necessary. There are many ways to develop the nonlinear preconditioner G, for example, the Jacobian of lower-order discretization, the domain decomposition preconditioners composed of Jacobian blocks on sub-domains of the of the full problem domain, and the inverse of the high-order term in the non-linear operator. In our application, we will use Nonlinear additive schwarz algorithm as preconditioner.

Nonlinear Additive Schwarz preconditioner

Naturally, the preconditioner should be close to the Jacobian inverse. Domain decomposition preconditioners [18] are based on approximating the high-order term (or the whole operator), subdividing the geometrical domain of the differential operator, computing the inverses on subdomains, and combining these inverses.

Let R_i, the restriction operator to the subdomain Ω_i, $i = 1, \ldots, Ns$, where Ns is the subdomains number. The subdomain nonlinear function F_i is then defined as :

$$F_i = R_i F.$$

For any $x \in \mathbb{R}^N$, let $T_i(x) : \mathbb{R}^N \to \mathbb{R}^N$ defined as the solution of the following subspace nonlinear systems :

$$F_i(x - R_i^T T_i(x)) = 0, \;\; \text{for} \;\; i = 1, \ldots, Ns. \tag{6}$$

The new global nonlinear function is hence defined as :

$$\mathcal{F}(x) = \sum_{i=1}^{Ns} R_i^T T_i(x), \tag{7}$$

which we refer to as the nonlinearly preconditioned $F(x)$ (for more details, see). In our numerical tests we will use the block Jacobi preconditioner which can be regarded as a zero-overlap form of additive schwarz [10].

3. Numerical results

An original image u_0 is noised with gaussian noise with variance of 7%. In figure (1(a)) the original image is shown, while the noisy image is shown in figure (1(b)).
The noisy image is processed by applying the evolution model as in [3] where the diffu-

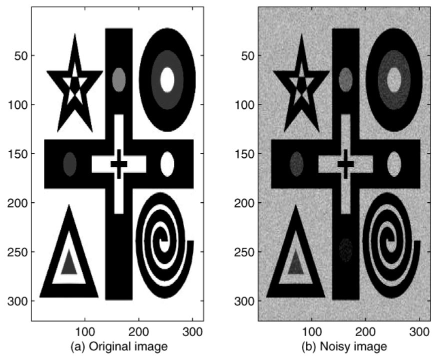

Figure 1. *Original and noisy images.*

sivity function is given by (2). We take $\alpha = 1E - 7$ and $\lambda = 1$. The partial differential equation is discretized an implicit finite differences scheme. The discrete nonlinear system is solved by a Newton-like methods.
Figure 2 shows the processed image in different time, where the step time is $dt = 1$. The nonlinear solver used is Newton-GMRES globalized by a Backtracking strategy.
Among the quantitative criteria most current to evaluate the performances of a denoising algorithm, we retained the Signal to Noise Ration (SNR), it is expressed in decibels by the relation between the image of reference I_1 and the image I_2 after analysis : $SNR(I_1/I_2) = 10 \log_{10} \left[\frac{\sigma^2(I_1)}{\sigma^2(I_1 - I_2)} \right]$, where σ is the variance [4]. Table 1 shows the SNR for different times.

In this example, the Newton-GMRES method converges more fast than the Broyden method. This last requires more function evaluations, for each time step, due to the Back-

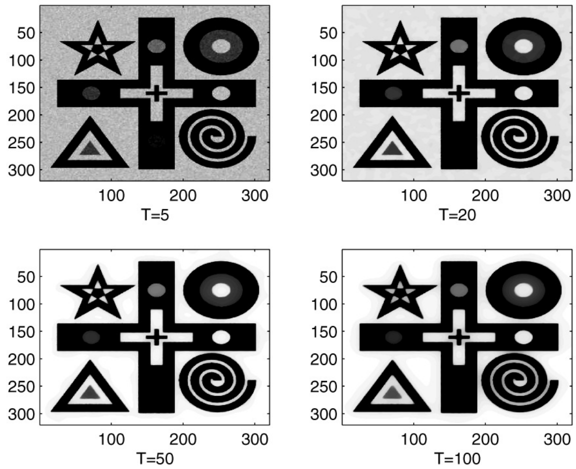

Figure 2. *Denoisy image .*

tracking reductions (table 2).

4. Conclusion

We presented methods to solve a nonlinear model in image processing discretized by an implicit finite differences scheme. Newton like methods with globalization are used. To accelerate the convergence and avoid failure in these methods, nonlinearly preconditioner are introduced. We applied this approaches to simple examples, and application to our model in image processing is in progress. Another future work is to use trust region methods to solve this nonlinear system.

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T	5	20	50	100
SNR	20.79	26.51	22.67	18.03

Tableau 1. *SNR for different times*

	CPU time (s)	Number of func. evals	Time of func. evals (s)
Newton-GMRES	623.593	748	587.263
Broyden	821.171	1047	759.705

Tableau 2. *CPU time required to solve the diffusion equation*

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