A simple adaptive observer for a class of Continuous linear time varying system with discrete output

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RÉSUMÉ. Dans ce travail préliminaire, nous proposons la construisons d'un observateur adaptatif pour un système continu-discret. Il est bien connu que les observateurs adaptatifs jouent un rôle très important dans l'estimation des états et des paramètres. Beaucoup de résultats sont connus sur les observateurs adaptatifs pour des systèmes continus à sortie continue (voir par exemple [1, 3, 7, 11]). dans ce travail nous considerons un système linéaire avec une une sortie discrète et nous proposons un algorithme pour estimer et l'état et le paramètre sans la condition d'exitation persistante. l'observateur adaptatif proposé est assez interessant du fait l'exponentielle convergence de l'erreur.

ABSTRACT. In this preliminary work an adaptive observer for continuous-discrete systems is proposed. It is well known that adaptive observer is very important for state and parameters estimation; most of known results concern continuous systems with continuous-output or discrete-time systems with discrete outputs (see for instance [1, 3, 7, 11]). In this paper we consider a linear system with discrete outputs and we propose an algorithm to estimate both the state and the parameter without any persistence condition. The proposed adaptive observer is shown to be quite promising due to the exponential error convergence.

MOTS-CLÉS : Observateur adaptatif, Estimation d'état et de paramèmtres, Systèmes continu-discret.

KEYWORDS : Adaptive observer, State estimation, parameters estimation, Continuous-discrete systems.

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1. Introduction

Since the seventies, many works have been done for adaptive observer (continuous systems with continuous outputs, discrete systems with discrete outputs), we can find somme of these works in [3, 6, 9] and references therein.
We consider here a linear system coupled with a discrete output y_k which is measured at times $0, 1, \dots, k, k+1, \dots$:

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + \varphi(t)\theta \\ y(k) = Cx(k) \end{array} \right. \tag{1}$$

This system is in its adaptive form (see [8] for instance) where the control is null. So A is an anti-shift matrix, $\varphi(t) \in \mathbb{R}^n$ is a time-varying vector, and θ is a constant real parameter. The observation matrix is given by $C = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}$. The goal of this paper is to derive a tool that would allow giving a dynamical estimation of the unknown state as well as the unknown parameter θ. In [1, 11] the same problem has been considered for continuous system with continuous output, and in [3] the dynamics as well as the output are discrete. In these papers an adaptive observer is built. This observer allows estimating the state and it also allows estimating the parameter if a certain persistence condition is satisfied. This condition is not easy to check for a given system.
In this paper we deal with a continuous system whose output is discrete and we propose an algorithm to estimate the state x and the parameter θ. This will be done without any persistence condition.

2. Main result

We consider the following candidate adaptive observer for system (1)

$$\left\{ \begin{array}{ll} \dot{z}(t) = Az(t) + \varphi(t)\hat{\theta}_k & t \in [k, k+1[, \\ z_{k+1} = z_{k+1}^- - K(Cz_{k+1}^- - y_{k+1}), & \\ \hat{\theta}_{k+1} = \hat{\theta}_k - \Gamma(Cz_k - y_k). & \end{array} \right. \tag{2}$$

Where :

- $z_{k+1}^- = e^A z_k + \hat{\theta} e^A \int_k^{k+1} e^{(k-s)A}\varphi(s)ds.$
- $\Gamma \in \mathbb{R}$ and $K \in \mathbb{R}^n$ are to be determined.

We shall use the notation $H_k = \int_k^{k+1} e^{(k-s)A}\varphi(s)ds$ so we can write $z_{k+1}^- = e^A z_k + \hat{\theta} e^A H_k$.

2.1. A stability result for almost autonomous discrete systems

To prove that system (2) is an adaptive observer for system (1) we need first to prove the following stability result whose continuous version can be found in [2] (pages 48-49) :

Proposition 2.1 *Consider a nonautonomous discrete time system that can be written*

$$v_{k+1} = (U + W(k))v_k, \tag{3}$$

where U is a constant matrix and $W(k)$ is a time-varying matrix. Suppose that all the eigenvalues of the matrix U belong to open unit disk of complex space $\mathbb{C}$, that is the *system $v_{k+1} = Uv_k$ is exponentially stable. Let $M > 0$ be such that $\|U^k\| \leq M\lambda^k$ with $M \in \mathbb{R}^+$ and $\lambda \in]0, 1[$. Then*

(i) *if there exists b_0 satisfying $b_0 < \dfrac{1-\lambda}{M}$ and $\|W(k)\| \leq b_0$ for all $k \geq k_0$, for some $k_0 \in \mathbb{N}$,*

or

(ii) *if there exists $b_1 > 0$ such that $\sum_{k_0}^{\infty} \|W(k)\| \leq b_1$,*

then the system (3) is exponentially stable.

Proof of proposition 2.1 For the proof of this proposition we use the following discrete version of Grönwall's Lemma (see for instance [10], page 198).

Lemma 2.1 *(discrete version of Gronwall inequality) Let z_n and h_n two real series, with $n \geq n_0 \geq 0$ and $h_n \geq 0$. If*

$$z_n \leq M\left[z_{n_0} + \sum_{j=n_0}^{n-1} h_j z_j\right], \qquad \text{for some } M > 0$$

then

$$z_n \leq z_{n_0} \prod_{j=n_0}^{n-1} [1 + Mh_j], \qquad n \geq n_0.$$

$$z_n \leq z_{n_0} exp\left[\sum_{j=n_0}^{n-1} Mh_j\right], \qquad n \geq n_0.$$

Going back to the Proposition 2.1, the solutions of (3) are of the form :

$$v_k = U^k v_0 + \sum_{i=0}^{k-1} U^{k-1-i} W(i) v_i$$

so

$$\begin{aligned} \|v_k\| &\leq \|U^k\|\|v_0\| + \sum_{i=0}^{k-1} \|U^{k-1-i}\|\|W(i)\|\|v_i\| \\ \|v_k\| &\leq \|U\|^k\|v_0\| + \sum_{i=0}^{k-1} \|U\|^{k-1-i}\|W(i)\|\|v_i\| \end{aligned}$$

The matrix U is exponentially stable, so there exist $M > 0$ and $\lambda \in [0, 1[$ such that $\|U^k\| \leq M\lambda^k, \forall k \geq k_0 > 0$

so we have :

$$\|v_k\| \leq M\lambda^k \left[\|v_0\| + \sum_{i=0}^{k-1} \lambda^{-r-1}\|W(i)\|\|v_i\|\right]$$

by a change of variable : $y_k = \lambda^{-k}\|v_k\|$, we have :

$$y_k \leq M\left[y_0 + \sum_{i=0}^{k-1} \lambda^{-1}\|W(i)\| y_i\right]$$

• Assuming that $\|W(i)\| \leq b_0, \forall i \geq 0$ and applying Lemma 2.1 we obtain :

$y_k \leq y_0 \prod_{i=0}^{k-1}(1 + M\lambda^{-1}b_0) = y_0(1 + M\lambda^{-1}b_0)^k$,

so $\|v_k\| \leq (\lambda + Mb_0)^k \|v_0\|$.
To have exponential stability of $(v_k)_k$, b_0 has to satisfy the following condition :

$$b_0 < \frac{1-\lambda}{M}$$

• Assuming now $\sum_{i=0}^{k-1} \|W(i)\| \leq b_1$ and applying Lemma 2.1 we obtain also :
$y_k \leq y_0 \exp(\lambda^{-1} \sum_{i=0}^{k-1} \|W(i)\|)$ and so :
$\|v_k\| \leq B\lambda^k \|v_0\|$, with $B > 0$; and this leads to the exponential stability of $(v_k)_{k\geq 0}$. ■

2.2. Observer convergence

Now we go back to the systems (1,2). Let us consider the state estimation error $e = z - x$ and the parameter estimation error $\tilde{\theta} = \hat{\theta} - \theta$. for $t \in [k, k+1[$, we have :

$$\begin{array}{rcl} \dot{e} & = & Az + \varphi\hat{\theta}_k - Ax - \varphi\theta \\ & = & Ae + \varphi(\hat{\theta}_k - \theta) \\ & = & Ae + \varphi\tilde{\theta}_k \end{array}$$

for $t = k+1$ we have :

$$\begin{array}{rcl} e_{k+1} & = & z_{k+1} - x_{k+1} \\ & = & z_{k+1}^{-} - KC(z_{k+1}^{-} - x_{k+1}) - x_{k+1} \\ & = & (I - KC)e_{k+1}^{-} \\ \text{and} & & \\ \tilde{\theta}_{k+1} & = & \tilde{\theta}_k - \Gamma C e_k. \end{array}$$

where $e_{k+1}^{-} = e^A e_k + e^A H_k \tilde{\theta}_k$

Let us now consider the dynamic of the error on the extremity of each interval $[k, k+1[$, we have :

$$\left\{ \begin{array}{l} e_{k+1} = (I - KC)e^A e_k + (I - KC)e^A H_k \tilde{\theta}_k \\ \tilde{\theta}_{k+1} = -\Gamma C e_k + \tilde{\theta}_k. \end{array} \right. \tag{4}$$

The system (4) can be writen in a matrix form as follows :

$$v_{k+1} = \begin{pmatrix} (I - KC)e^A & (I - KC)e^A H_k \\ -\Gamma C & 1 \end{pmatrix} v_k,$$

where $v_k = \begin{pmatrix} e_k \\ \tilde{\theta}_k \end{pmatrix}$.
Therefore the error equation (4) has the same form as system (3) of Proposition 2.1 :

$$v_{k+1} = (U + W(k))v_k, \tag{5}$$

A possible choice for the matrices U and $W(k)$ is : $U = \begin{pmatrix} (I - KC)e^A & O \\ -\Gamma C & 1 \end{pmatrix}$ and $W(k) = \begin{pmatrix} 0 & (I - KC)e^A H_k \\ 0 & 0 \end{pmatrix}$. But with this choice we can not use Proposition 2.1

because 1 is an eigenvalue of the matrix U whatever what the values of Γ and K are. Thus we are going to choose :

$$U = \begin{pmatrix} (I-KC)e^A & (I-KC)e^A H_0 \\ -\Gamma C & 1 \end{pmatrix}$$
$$W(k) = \begin{pmatrix} 0 & (I-KC)e^A(H_k - H_0) \\ 0 & 0 \end{pmatrix}.$$

We can now formulate our result :

Proposition 2.2 *If there exist $\Gamma \in \mathbb{R}$ and $K \in \mathbb{R}^n$ in such a way that the matrix U has all its eigenvalues with modulus strictly less than 1 and if $W(k)$ satisfies the condition* **(i)** *or* **(ii)** *of Proposition 2.1, then the system (2) is an adaptive observer for system (1), that is $z_k - x_k$ tends to 0 and θ_k tends to θ as k tends to $+\infty$.*

It is actually possible to find $\Gamma \in \mathbb{R}$ and $K \in \mathbb{R}^n$ in such a way that the matrix U is stable but the proof of this fact is quite long. For lack of space we detail the construction of Γ and K for $n = 2$, i.e., when the system (1) is defined on $\mathbb{R}^2$.

3. Study of a particular case : $n = 2$

For $n = 2$, we prove the existence of Γ and K such that the eigenvalues of U belong to the open unit disk of $\mathbb{C}$. Here we consider :
$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = [0,1]$, $K = \begin{pmatrix} K1 \\ K2 \end{pmatrix}$, $\varphi(t) = \begin{pmatrix} \varphi_1(t) \\ \varphi_2(t) \end{pmatrix}$ and we denote $H_0 = \begin{pmatrix} a \\ b \end{pmatrix}$.

Thus :
$v_{k+1} = (U + W(k))v_k$
where

$$U = \begin{pmatrix} 1-K_1 & -K_1 & (1-K_1)a - K_1 b \\ 1-K_2 & 1-K_2 & (1-K_2)(a+b) \\ 0 & -\Gamma & 1 \end{pmatrix}$$
and $W(k) = \begin{pmatrix} 0 & (I-KC)e^A(H_k - H_0) \\ 0 & 0 \end{pmatrix}$.

Let us consider the matrix U, we have to choose the vector $\bar{K} = (K_1, K_2, 0)^T$ and Γ in order to make matrix U exponentially stable. For this, we write :

$U = \tilde{U} - \bar{K}\tilde{C}$, where $\tilde{C} = (1, 1, a+b)$ and $\tilde{U} = \begin{pmatrix} 1 & 0 & a \\ 1 & 1 & a+b \\ 0 & -\Gamma & 1 \end{pmatrix}$.

First we shall construct $\tilde{K} = (K_1, K_2, K_3)^T$ such that $\tilde{U} - \tilde{K}\tilde{C}$ has all its eigenvalues in the open unit disk and after we shall choose Γ in order to have $K_3 = 0$. A sufficient and necessary condition for the existence of such a $\tilde{K}$ is that the pair $(\tilde{U}, \tilde{C})$ is observable or $(\tilde{U}^T, \tilde{C}^T)$ is controllable.

Let $B = \tilde{C}^T$, $H_c = (B, \tilde{U}^T B, \tilde{U}^{T^2} B)$ then

$$H_c = \begin{pmatrix} 1 & 2 & 3 - \Gamma(a+b) \\ 1 & 1 - \Gamma(a+b) & 1 - 4\Gamma a - 3\Gamma b \\ a+b & 3a+2b & 5a+2b+(a+b)(1-\Gamma(a+b)) \end{pmatrix}$$

and $det(H_c) = -a(1 + b\Gamma)$.
$(\tilde{U}^T, \tilde{C}^T)$ is controllable if and only if $a(1 + b\Gamma) \neq 0$.
a and b are fixed constants, so it is always possible to choose Γ such that to satisfy the condition of controllability if $a \neq 0$.
Let us fix $\lambda_1, \lambda_2, \lambda_3$ as the desired spectrum of the matrix $\tilde{U}$ and let $Q(x)$ be the corresponding characteristic polynomial. By using Ackermann formula to compute $\tilde{K} = \left(\tilde{K}_1, \tilde{K}_2, \tilde{K}_3\right)$ which gives

$$\tilde{K}_1 = \frac{\Gamma\lambda_1\lambda_2\lambda_3 a^2 + (b\Gamma+1)(-\lambda_2\lambda_3 + \lambda_1(\lambda_2(2\lambda_3-1)-\lambda_3)+1)a + b(b\Gamma+1)(\lambda_1-1)(\lambda_2-1)(\lambda_3-1)}{b\Gamma a + a}$$

$$\tilde{K}_2 = \frac{b\Gamma - \lambda_1\lambda_2\lambda_3 + 1}{b\Gamma+1}$$

$$\tilde{K}_3 = \frac{-b\Gamma(\lambda_1-1)(\lambda_2-1)(\lambda_3-1) + (\lambda_2-1)(\lambda_3-1) + \lambda_1(\lambda_3 - \lambda_2(a\Gamma\lambda_3 + \lambda_3 - 1) - 1)}{b\Gamma a + a}.$$

Therefore we choose $\Gamma = \frac{(\lambda_1-1)(\lambda_2-1)(\lambda_3-1)}{-b(\lambda_1-1)(\lambda_2-1)(\lambda_3-1) + a\lambda_1\lambda_2\lambda_3} = \dfrac{Q(1)}{-bQ(1) + a\lambda_1\lambda_2\lambda_3}$, this choice of Γ leads to $\tilde{K}_3 = 0$. ■

4. Numerical example and simulation

Let us consider the system (1) defined on $\mathbb{R}^2$ with $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $C = [0, 1]$, $\varphi(t) = \begin{pmatrix} \dfrac{t+1}{t+2} \\ 0 \end{pmatrix}$, We compute $H_0 = \begin{pmatrix} 1 - \ln \dfrac{3}{2} \\ \dfrac{1}{2} - \ln \dfrac{9}{4} \end{pmatrix}$, and

$$W(k) = \begin{pmatrix} 0 & 0 & (1 - K1)(\ln \frac{3}{2} - \ln \frac{k+3}{k+2}) - K1(\ln \frac{9}{4} - (k+2) \ln \frac{k+3}{k+2} \\ 0 & 0 & (1 - K2)(3 \ln \frac{3}{2} - (k+3) \ln \frac{k+3}{k+2}) \\ 0 & 0 & 0 \end{pmatrix}.$$

We use the following matrix norm : $\|W(k)\| = [\rho(W^T W)]^{1/2}$.
and we choose $\lambda_1 = \dfrac{1}{2}$; $\lambda_2 = \dfrac{3}{5}$ and $\lambda_3 = \dfrac{1}{5}$ where λ_i, $(i = 1, 2, 3)$ are the desired eigenvalues for U. With these values, we compute : $K = \begin{pmatrix} 0.8436769 \\ 0.8563231 \end{pmatrix}$ and $\Gamma = 18.730773$. These give the following simulation, the real value of the parameter θ is 5, and we can see in our simulation graphic (Figure 2) that θ_k converges to 5. Figure 1 shows the convergence of the state estimation $z_1(k)$ towards $x_1(k)$. The same curves are obtained for the second component of the state and its estimation.

5. Conclusion

In this work, we construct an adaptive observer that allows a state estimation as well as a parameter estimation for linear time varying system. The construction does not depend on any persistence condition which is always difficult to check. For this first work, we simplify the system in order to propose an explicit algorithm for adaptive observer with discrete output. We prospect to extend this result for a more general case.

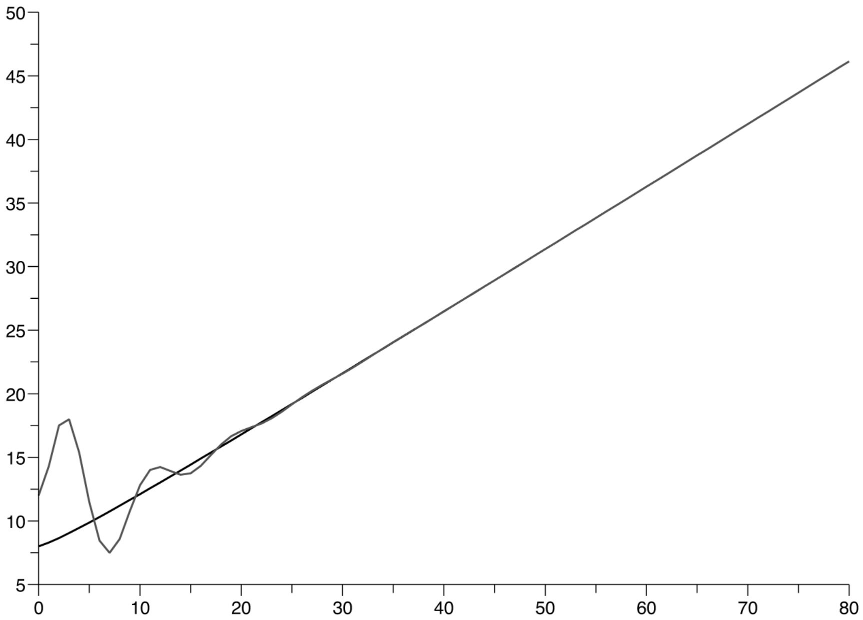

Figure 1 – *State X1 and its estimation*

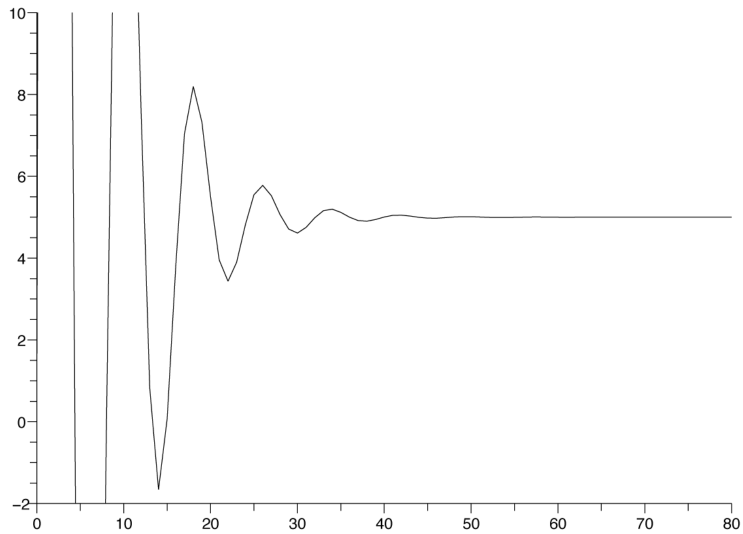

Figure 2 – *Estimation of parameter θ*

6. Bibliographie

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