Identification of small inclusions from multistatic data using the reciprocity gap concept

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Abstract. We are interested in the identification of dielectric inclusions embedded into a homogeneous medium from measurements of the total electromagnetic Cauchy data (i.e. tangential electric and magnetic fields) on the boundary of this medium at a fixed frequency and for several incident waves. We shall consider the cases where the size of the inclusions is small compared to the wavelength. This configuration is typically the case, for instance, of imaging experiments based on microwaves used to detect malignancies or tumors.

We propose here the use of an algorithm that combines ideas from samplings methods (Linear Sampling, Factorization method, MUSIC) with the reciprocity gap concept and that was first introduced by Colton-Haddar [3] for the reconstruction of extended targets. The main advantage of this algorithm is that it does not require the evaluation the full background Green tensor (as required by the Linear Sampling method), nor the subtraction of the Dirichlet to Neumann operator of the medium (as required by the Factorization method). The Dirichlet to Neumann operator is implicitly removed when evaluating the reciprocity gap functional.

The main focuses of our contribution are:

- Indicate how one can remove the approximation argument used in the case of extended targets when the small inclusion asymptotic regime is valid: one obtains an exact characterization of the location of small objects from the range of constructed sampling operator.
- Identify some physical properties of the dielectric (those contained in the amplitude of the first asymptotic term) from the indicator function and the used family of parametrization (single layer potentials, Herglotz waves, ...).

We shall restrict ourselves to a 2-D setting of the problem and consider the cases of both monopole-source and dipole-source asymptotic behaviors. Numerical tests are given to demonstrate the efficiency of the algorithm and evaluate its resolution and robustness with respect to the measurement noise. We shall also discuss the validity of our approach with respect to the dielectric sizes.

1. Introduction

We consider a bounded domain Ω of IR^2 with sufficiently smooth boundary Γ, the place where we will carry out the measures, holding a homogeneous environment of electrical permittivity ϵ_0 and magnetic permeability μ_0 with waves number k which can be complex for an absorbent environment, containing m small inclusions $D_j^\alpha = z_j + \alpha B_j$ where z_j represent the center, α the size of the inclusion that we will tends to 0 and B_j is a bounded smooth domain ($\mathcal{C}^\infty$) containing the origin with electrical permittivity ϵ_j and magnetic permeability μ_j for $j = 1, ..., m$. If u^i is an incident point source located in $x_0 \in \Sigma$ where Σ is a boundary of $IR^2 \setminus \bar{\Omega}$, $u^i(x) = \Phi(x_0, x)$ where $\Phi(x_0, y) = \frac{i}{4} H_0^1(k|x_0 - y|)$ is the Green function solution of

$$\begin{cases} \Delta_y \Phi(x_0, y) + k^2 \Phi(x_0, y) = -\delta_{x_0}(y) & \text{dans} \quad IR^2 \\ \lim\limits_{r=|x| \to \infty} r^{\frac{1}{2}} (\dfrac{\partial \Phi}{\partial r} - ik\Phi) = 0 \end{cases} \tag{1}$$

where $H_0^{(1)}$ is the Hankel function of first kind of order 0.
Then the direct diffraction problem is to find $u_\alpha \in H^1_{loc}(IR^2)$ solution of

$$\nabla(\cdot \frac{1}{\mu_\alpha} \nabla u_\alpha) + w^2 \epsilon_\alpha u_\alpha = 0 \quad \text{dans} \quad IR^2 \tag{2}$$

$$\lim_{r=|x| \to \infty} r^{\frac{1}{2}} (\frac{\partial u_\alpha^s}{\partial r} - iku_\alpha^s) = 0 \tag{3}$$

Where

$$u_\alpha = u^i + u_\alpha^s \tag{4}$$

and μ_α is the magnetic permeability function, piecewise constant defined by:

$$\mu_\alpha(x) = \begin{cases} \mu_0 & si \quad x \in IR^2 \setminus \cup_{j=1}^m D_j^\alpha \\ \mu_j & si \quad x \in D_j^\alpha, \ j = 1, ..., m \end{cases} \tag{5}$$

and ϵ_α is the electric permittivity function piecewise constant defined by:

$$\epsilon_\alpha(x) = \begin{cases} \epsilon_0 & si \quad x \in IR^2 \setminus \cup_{j=1}^m D_j^\alpha \\ \epsilon_j & si \quad x \in D_j^\alpha, \ j = 1, ..., m \end{cases} \tag{6}$$

u_α^s is the scattering field, w is the frequency and k is the waves number in free space.
From [1] the scattering field u_α^s has the asymptotique behavior when the size α of the inclusions tends to 0:

$$u_\alpha^s = \alpha^2 \sum_{j=1}^m \left[\gamma_\mu^j \nabla \Phi(x, z_j) \cdot M^j(\frac{\mu_j}{\mu_0}) \nabla u^i(z_j) + |B_j| k^2 \gamma_\epsilon^j u^i(z_j) \Phi(x, z_j) \right] + o(\alpha^2) \tag{7}$$

Where $\gamma_\mu^j = \frac{\mu_j}{\mu_0} - 1$, $\gamma_\epsilon^j = \frac{\epsilon_j}{\epsilon_0} - 1$ and $M^j(\frac{\mu_j}{\mu_0})$ is the polarization tensor of B_j.
We ramark that when the inclusions have the same electric permittivities than Ω, they behave as a dipolar sources and when they have magnetic permeabilities than Ω, they behave like polar sources.

2. Identification Process

We assume in this paragraph that the inclusions $D_1^\alpha, D_2^\alpha, ..., D_m^\alpha$ are with respectively electrical permittivities $\epsilon_1, \epsilon_2, ..., \epsilon_m$ and with common magnetic permeability $\mu_j = \mu_0, \ j = 1, ..., m$. Let $u^i = \Phi(x_0, \cdot)$ an incident point source located in $x_0 \ \ in \ \Sigma$ where Σ is a boundary outside Ω. The inverse problem is to identify the centres z_j of the inclusions in Ω from the knowledge of the total field $u_\alpha = u^i + u_\alpha^s$ and its normal derivative field measured on the boundary $\partial\Omega$.
According to [1], when the size of inclusion is very small, the diffracted field u_α^s is approached by $\tilde{u}_\alpha^s$ in the following manner:

$$\tilde{u}_\alpha^s = \alpha^2 \sum_{j=1}^m |B_j| k^2 \gamma_\epsilon^j u^i(z_j) \Phi(x, z_j)$$

We denote by $\mathcal{U}$ the set of the approximate total field and his normal derivative when x_0 describes Σ.

$$\mathcal{U} = \left\{ \left(\tilde{u}_\alpha(x_0, \cdot), \frac{\partial \tilde{u}_\alpha}{\partial \nu}(x_0, \cdot) \right), x_0 \in \Sigma \right\}$$

ν is the outer unit normal of the boundary $\partial\Omega$ and by $\mathcal{V}$ the set of test functions solutions of Helmholtz equation in $\Omega \setminus \cup_{j=1}^{m} D_j^\alpha$

$$\mathcal{V} = \left\{ v/\Delta v + k^2 v = 0 \text{ dans } \Omega \setminus \cup_{j=1}^{m} D_j^\alpha \right\}$$

We define the reciprocity gap functional $\mathcal{R}$ by:

$$\mathcal{R}(u, v) = \int_{\partial\Omega} \left(u \frac{\partial v}{\partial \nu} - v \frac{\partial u}{\partial \nu} \right) \, ds \quad (u, v) \in \mathcal{U} \times \mathcal{V}$$

Let S the single-layer operator defined by:

$$S : L^2(\Sigma) \longrightarrow \mathcal{V} : \quad g \longmapsto Sg$$

where

$$Sg(x) = \int_\Sigma \Phi(x, y) \, g(y) \, ds(y) \text{ pour } x \in \Omega$$

We denote for $z \in \Omega$ and $x_0 \in \Sigma$

$$l_z(x_0) = \mathcal{R}(u(x_0, \cdot), \Phi(z, \cdot)) \tag{8}$$

Our problem consists for $z \in \Omega$ to find $g \in L^2(\Sigma)$ solution of:

$$\mathcal{R}(u(x_0, \cdot), Sg) = l_z(x_0) \ \forall \, x_0 \in \Sigma \tag{9}$$

We have the following result

Theorem 1 *The equation*

$$\mathcal{R}(u(x_0, \cdot), Sg) = l_z(x_0) \ \forall \, x_0 \in \Sigma \tag{10}$$

has a solution g if and only if $z \in \{z_j, j = 1, ..., m\}$

3. Numercal Results

In the previous paragraph we showed that the equation (9) has a solution if and only if z is the center of one of the inclusions located in Ω, we observe that if $z = z_{j_0}$ then g_z is proportional to h_z which is the orthogonal in $L^2(\Sigma)$ to the vectoriel subspace G spanned by $\Phi(z_j, .)$ $j \neq j_0$. More precisely,

$$g_{z_{j_0}} = \frac{1}{\lambda_{j_0}^\alpha} \frac{h_{z_{j_0}}}{\int_\Sigma \Phi(z_{j_0}, y) \, h_{z_{j_0}}(y) \, ds(y)}$$

otherwise $\|g_z\|_{L^2(\Sigma)} = +\infty$.

The linear sampling method consists to plot the contours of the function:

$$\begin{aligned} \Omega &\longrightarrow IR \\ z &\mapsto \frac{1}{\|g_z\|_{L^2(\Sigma)}} \end{aligned}$$

One observes peaks at the location of the small inclusions.

Remark 1 *The identification of positions of the point sources allows us to numerically calculate the intensity of this point, in fact, the numerical computation of $g_{z_{j_0}}$ leads to the determination of λ_{j_0} through the equality*

$$\int_\Sigma \Phi(z_{j_0}, y) \, g(y) \, ds(y) = \frac{1}{\lambda_{j_0}}$$

3.1. Identification of inclusions

Test on the index of medium

In this section, we test the influence of the inclusions index in our inverse problem of identification, Ω is a square centred in $(0,0)$ where the length of its side is 4λ with a fixed index $n_\Omega = 2$, this medium contains two circular inclusions D_1 and D_2 centred respectively in $z_1 = (0.5, 0)$ et $z_2 = (-0.5, 0)$ and with the same ray $r = 0.05$, we fixed the index n_1 of D_1, $n_1 = 3$ and we vary the index n_2 of D_2, $n_2 = 2.2,\ 2.4,\ 2.6,\ 2.8$ and 3, the number of the mesure points in the boundary of Ω is 240 points, the incident point sources number is 256 regularly ditributed in Σ wich is represented by 4 segments round of Γ and distant $\frac{\lambda}{2}$ from Γ where the wave length in the free space is $\lambda = 1$, the sampling points z of Ω are located in the unit square 40×40 points.

In the figures1,2 we plot the curve of the function defined by:

$$[-1,1] \longrightarrow IR$$

$$z = (x,0) \mapsto \frac{1}{\|g_z\|_{L^2(\Sigma)}}$$

For respectively noisy data 1% and 5% and we fixe for every curve the index of the inclusions. This curves show that if the index of the inclusion is greater than the index of the medium Ω then it's well detected and the size of the peaks are proportional to the index of the inclusions.

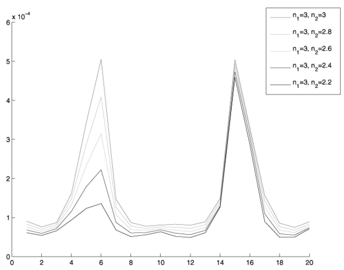

Figure 1. Test on the index , $n_\Omega = 2$, $n_1 = 3$, $n_2 = 2.2,\ 2.4,\ 2.6,\ 2.8,\ 3$, noise=1%.

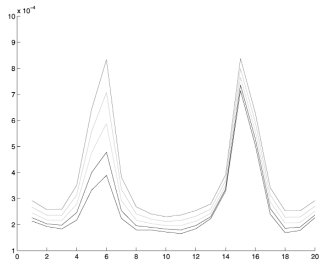

Figure 2. Test on the index , $n_\Omega = 2$, $n_1 = 3$, $n_2 = 2.2,\ 2.4,\ 2.6,\ 2.8,\ 3$, noise=5%.

In the figures 3,4 and 5, D_1 is an inclusion with perfect conductivity (hight index) and D_2 is an inclusion wich we vary its index: $n_2 = 3 + i,\ 3 + 2i$ and $4 + 5i$, the other parametre are like the precedent tests the noise of the data is equal to 1%, we show that the inclusion D_1 is well detected and when the inclusion D_2 becomes more absorbent its identification is more precisly. In the figures 3,4, 5 and in the following figures the red points represent the peaks of the function

$$\Omega \longrightarrow IR$$

$$z \mapsto \frac{1}{\|g_z\|_{L^2(\Sigma)}}$$

and this points represent the location of the small inclusion solution of our inverse problem.

Test on the size of inclusion

In figures 6,7 and 8 we test our algorithm when we vary the size of the inclusions. we consider two circular inclusions D_1 and D_2, where $z_1 = (0.5, 0)$, $z_2 = (-0.5, 0)$ and with same index

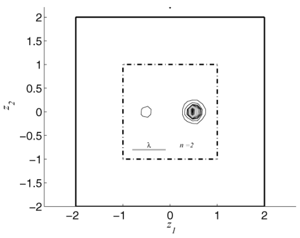

Figure 3. $n_\Omega = 2,\ n_2 = 3 + i,\ n_1$ =perfect inclusion.

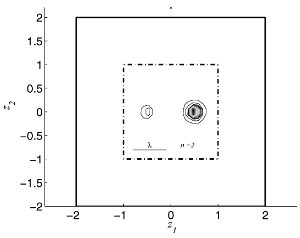

Figure 4. $n_\Omega = 2,\ n_2 = 3 + 2i,\ n_1$ =perfect inclusion.

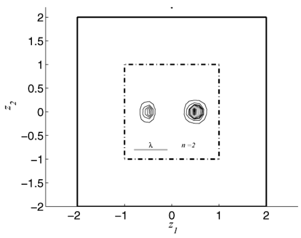

Figure 5. $n_\Omega = 2,\ n_2 = 4 + 5i,\ n_1$ =perfect inclusion

$n_1 = n_2 = 3$, we fixe the ray of D_1 $r_1 = 0.005$ and we vary the ray of D_2 $r_2 = 0.002,\ 0.003,\ 0.004$, we keep the other parametre of these tests like the precedent tests the noise of the data is equal to 1%, we remark that when the size of the inclusion is very small we can't detect it, it's considered like a noise for our algorithm.

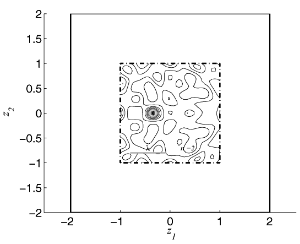

Figure 6. Influence de la taille de l'inclusion, $n_\Omega = 2,\ n_1 = n_2 = 3,\ r_1 = 0.005,\ r_2 = 0.002$.

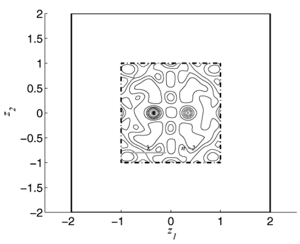

Figure 7. $n_\Omega = 2,\ n_1 = n_2 = 3,\ r_1 = 0.005,\ r_2 = 0.003$.

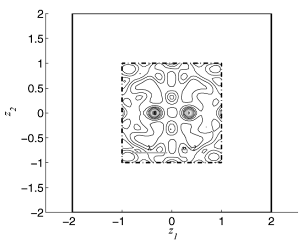

Figure 8. $n_\Omega = 2,\ n_1 = n_2 = 3,\ r_1 = 0.005,\ r_2 = 0.004$.

Polar sources

In figure 9 Ω is a square centred in $(0,0)$ where the length of its side is 4λ with a fixed index $n_\Omega = 1$, this medium contains two polar sources $S_1(0.5, 0.5)$ and $S_2(0, 0.5)$ distant of $\frac{\lambda}{2}$ and with the same intensity. we generate syntheticly the Cauchy data in the boundary evaluate in 400 points of mesures. the incident field $u^i = \Phi(x_0, .)$ and the scattering field $u^s(x_0, .) = \sum_{j=1}^{m} \lambda_j\ u^i(x_0, S_j)\ u^i(S_j, .)$, the number of x_0 is 256 wich are regularly ditributed in Σ wich is represented by 4 segments round of Γ and distant $\frac{\lambda}{2}$ from Γ, the linear sampling method gives us a good result.

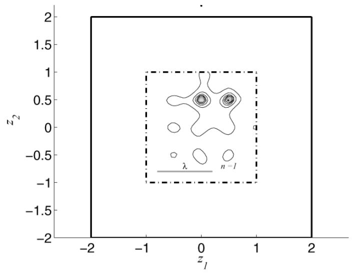

Figure 9. $d(S_1, S_2) = \frac{\lambda}{2}$

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