Adaptive Algorithm for Chorin's scheme

Application to the linearized Navier-Stokes Equations

Nizar Kharrat* Zoubida Mghazli**

* ENIT–LAMSIN
Université Tunis El Manar B.P.37, 1002 Tunis-Belvédère
Tunisie
nizar.kharrat@enit.rnu.tn
**E.I.M.A.
Université Ibn Tofail, Faculté des Sciences, Equipe d' Ingénierie Mathématique
B.P. 133 Kenitra
MAROC
mghazli_zoubida@yahoo.com

RÉSUMÉ. Les équations de Stokes instationnaires sont discrétisées par le schéma de projection de Chorin en utilisant la méthode des éléments finis de Galerkin continue en temps et en espace. Dans le but de construire un algorithme adaptatif combinant l'approximation en temps et en espace, nous développons des estimateurs de l'erreur résiduelle.

ABSTRACT. The time-dependent Stokes equations are discretized with Chorin's projection scheme by using continuous Galerkin finite element method in time and space. In order to built an adaptive algorithm that combines time and space approximation, we derive residual error estimators.

MOTS-CLÉS : équations de Stokes, schéma de projection de Chorin, éléments finis de Galerkin, estimateurs de l'erreur résiduelle.

KEYWORDS : Stokes equations, Chorin's projection scheme, Galerkin finite element, residual error estimators.

1. Introduction

Let Ω be a bounded connected domain of $\mathbf{R}^d$ $(d = 2, 3)$, with Lipschitz continuous boundary Γ. We consider the time-dependent Stokes problem in the primitive variables

$$\begin{cases} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega \times]0, T[, \\ \nabla . \mathbf{u} = 0 & \text{in} \quad \Omega \times]0, T[, \\ \mathbf{u} = 0 & \text{on} \quad \Gamma \times]0, T[, \\ \mathbf{u}_{|t=0} = \mathbf{u}_0 & \text{in} \quad \Omega. \end{cases} \tag{1}$$

The unknowns are the velocity $\mathbf{u} = \mathbf{u}(x, t)$ and the pressure $p = p(x, t)$; the data are $\mathbf{f} = \mathbf{f}(x, t)$ which represent a prescribed body force, and $\mathbf{u}_0 = \mathbf{u}_0(x)$ is the initial velocity, while ν is the kinematic viscosity assumed to be a positive constant. For the sake of simplicity, we consider a homogenous Dirichlet boundary condition.
In the late 1960s, the Chorin's scheme or projection method was introduced to overcome the numerical difficulty linked to the incompressibility constraint that relates both velocity and pressure. The interesting feature of this method, consists in solving at each time step, a sequence of uncoupled elliptic problems for velocity and pressure. However, this splitting involves an error that is essentially caused by the non-physical boundary condition on the pressure, which prevents the scheme to have optimal convergence behavior. In this context, the error analysis given by Rannacher [9] and that by Prohl [8], show a first order rate of convergence in time for the velocity with only a half order for the pressure in the L^2-norm. The authors had indeed conjectured that, a first order rate of convergence for the pressure is ibtained at the interior of Ω, by revealing the existence of boundary layers with a prescribed thickness on Γ.
Our purpose is an attempt for a best convergence rate on both pressure and velocity in appropriate norms. To this end, we perform a posteriori analysis on time and space discretization errors induced by Chorin's scheme. In this context, a first analysis of the spatial error has been carried out in [7], by using a continuous finite-element approach. Here, we complete the previous analysis by taking into account the time error in the framework of a continuous and piecewise affine approach. The key observation in the present analysis, consists in re-interpreting the projection method as a pressure stabilization method, where the time-step is viewed as a perturbation parameter. For this, we consider a single representation of the velocity, sought at the prediction step which stands for the diffusion part of the Stokes equations.
Using the standard residual technique and by separating the errors due to time and space discretization [1, 3], we derive two distinct family of time estimators. They are respectively defined in terms of the discrete velocity and pressure. In particular, we identify the first family with the estimators of [3], obtained for the backward Euler scheme. In addition, they can be defined independently of the spatial discretization method. We note that they are local with respect to time-step, and global with respect to space variables.
Then, we present how the error is bounded from above and from below by the Hilbertian sum of such estimators up to some terms involving the data. Finally, in order to control this error, we propose a general algorithm that combines adaptation between time step and mesh size.

2. The projection scheme

2.1. Preliminaries and notations

For convenience, we make use of the following notations :

$$X = H_0^1(\Omega)^d, \quad X' = H^{-1}(\Omega)^d, \quad Y = L^2(\Omega)^d \quad \text{and} \quad M = H^1(\Omega) \cap L_0^2(\Omega),$$

where $L_0^2(\Omega)$ denotes the subspace of $L^2(\Omega)$ with zero mean value on Ω. As usual, $H_0^1(\Omega)$ is equipped with the semi-norm $|\cdot|_1$ of $H^1(\Omega)$ and $L^2(\Omega)$ with the norm $\|\cdot\|_0$. In particular, $|\cdot|_1$ is a norm on M. We introduce the subspaces,

$$\mathbf{V} = \{\mathbf{v} \in X;\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\} \quad \text{and} \quad \mathbf{H} = \{\mathbf{v} \in Y;\ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v}.\vec{n} = 0 \text{ on } \Gamma\}.$$

We note the continuous and dense imbeddings $\mathbf{V} \subset \mathbf{H} = \mathbf{H}' \subset \mathbf{V}'$, where $\mathbf{V}'$ and $\mathbf{H}'$ are respectively the duals of $\mathbf{V}$ and $\mathbf{H}$. For any t, $0 \le t \le T$, we define the energy norm on $L^2(0,t;X) \cap \mathcal{C}^0(0,t;Y)$ by

$$[\mathbf{v}](t) = \left(\|\mathbf{v}(t)\|_0^2 + \nu \int_0^t |\mathbf{v}(s)|_1^2 ds \right)^{1/2}.$$

In the following, we assume Ω satisfying the H^2-ellipticity property, which implies that the solution of the stationnary Stokes problem with homogeneous Dirichlet data on Γ to be regular. Furthermore, if the data $(\mathbf{u}_0, \mathbf{f})$ are given into $\mathbf{V} \times L^2(0,T;Y) \cap \mathcal{C}^0(0,T;X')$, then problem (1) has a unique solution [2], such that

$$(\mathbf{u}, p) \in L^2(0,T; H^2(\Omega)^d) \cap H^1(0,T;Y) \times L^2(0,T;M), \quad \partial_t \mathbf{u} \in L^2(0,T;\mathbf{H}).$$

and for all $t \in [0,T]$,

$$\|\partial_t \mathbf{u}\|_{L^2(0,t;\mathbf{V}')} \le 2\|\mathbf{f}\|_{L^2(0,t;X')} + \nu^{\frac{1}{2}} \|\mathbf{u}_0\|_0 \text{ and } [\mathbf{u}](t) \le \left(\nu^{-1} \|\mathbf{f}\|_{L^2(0,t;X')}^2 + \|\mathbf{u}_0\|_0^2 \right)^{\frac{1}{2}}.$$

2.2. The semi-discrete version

Let N be an integer and $0 = t_0 < t_1 < \cdots < t_N = T$ a partition of $[0,T]$, with step size $\tau_n = t_n - t_{n-1}$ such that the parameter $\sigma_\tau = \max\limits_{2 \le n \le N} \dfrac{\tau_n}{\tau_{n-1}}$ is bounded.
To approximately solve (1) at time t_n in the framework of time adaptability, the Chorin's algorithm splits each iteration step into two parts. Start with $\mathbf{u}^0 = \mathbf{u}(0)$ and given $\mathbf{u}^{n-1}$, we look for a provisional velocity $\tilde{\mathbf{u}}^n$ in X, solution to

$$\begin{cases} \dfrac{\tilde{\mathbf{u}}^n - \mathbf{u}^{n-1}}{\tau_n} - \nu \Delta \tilde{\mathbf{u}}^n = \mathbf{f}^n & \text{in} \quad \Omega, \\ \tilde{\mathbf{u}}^n = 0 & \text{on} \quad \Gamma, \end{cases} \tag{2}$$

then, we search for a correction $\mathbf{u}^n$ in $\mathbf{H}$ and a pressure Φ^n in M, solutions to the Darcy problem

$$\begin{cases} \dfrac{\mathbf{u}^n - \tilde{\mathbf{u}}^n}{\tau_n} + \nabla \Phi^n = 0 & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u}^n = 0 & \text{in} \quad \Omega, \\ \mathbf{u}^n \cdot \vec{n} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{3}$$

where $\vec{n}$ denote the normal unit vector external to Γ and $\mathbf{f}^n$ an approximation of $\mathbf{f}(t_n)$. The sequence Φ^n is also a solution to a homogeneous Poisson-Neumann problem. In practice, the inconsistent boundary condition satisfied by Φ^n leads to boundary layers, which prevents the scheme to be a first order in time for the pressure. However, despite this incompatibility condition, $(\tilde{\mathbf{u}}^n, \Phi^n)$ (or $(\mathbf{u}^n, \Phi^n)$) remains an admissible appproximation to the solution of (1) in appropriate norms [9].

2.3. The fully-discrete version

For simplicity, we assume that Ω is a polygone (d=2) or a polyhedron (d=3). Then, as we intend to use mesh adaptivity at each time step, we denote by $\{\mathcal{T}_h^n\}_{0\le n\le N}$ the triangulations of Ω, made up triangles (d=2) or tetrahedra (d=3), with maximal diameter h. These triangulations are assumed to have the regularity properties established in [11]. Next, in the framework of a continuous Galerkin finite element approximation, we consider the sequence of subspaces $(X_h^n, Y_h^n, M_h^n)_{0\le n\le N}$ built over the meshes $\{\mathcal{T}_h^n\}_{0\le n\le N}$ and contained in (X, Y, M). Now, the discrete counterpart of (2)-(3) read as follow :
given one approximation $\mathbf{u}_h^0$ of $\mathbf{u}_0$, find $(\tilde{\mathbf{u}}_h^n, \mathbf{u}_h^n, \Phi_h^n)_{1\le n\le N}$ in $(X_h^n \times Y_h^n \times M_h^n)^N$ such that, for $n = 1 \dots N$,

$$(\tilde{\mathbf{u}}_h^n, \mathbf{v}_h) + \nu\tau_n(\nabla\tilde{\mathbf{u}}_h^n, \nabla\mathbf{v}_h) = (\mathbf{u}_h^{n-1}, \mathbf{v}_h) + \tau_n\langle \mathbf{f}^n, \mathbf{v}_h\rangle_{X',X} \quad \forall \mathbf{v}_h \in X_h^n, \tag{4}$$

$$\begin{cases} (\mathbf{u}_h^n, \mathbf{v}_h) + \tau_n(\nabla\Phi_h^n, \mathbf{v}_h) = (\tilde{\mathbf{u}}_h^n, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in Y_h^n, \\ (\mathbf{u}_h^n, \nabla q_h) = 0 \quad \forall q_h \in M_h^n. \end{cases} \tag{5}$$

It is proved[1] that, for a convenient choice of Y_h^n, the unknown Φ_h^n is also solution to the discrete Poisson-Neumann problem and consequently the space Y_h^n become unusefull in practice. Thus, the main unknowns of the algorithm (4)-(5) will be then $(\tilde{\mathbf{u}}_h^n, \Phi_h^n)$. Furthermore, it is argued in [5] that is not permitted to use equal order polynomial interpolation for X_h^n and M_h^n which should satisfy the compatibility condition of Babùska-Brezzi.

3. Residual based error estimators

We present here, the estimators that will be used for the adaptation strategy to control both time and space discretization errors. Following the analysis of [1, 3], we exhibit two families of error estimators. The idea is based onto the decoupling between the time and space discretization error. In this way, we first derive the estimators linked with the time error between the solution of (1) and that of (2)-(3), then the estimators associated with the spatial error between the solution of (2)-(3) and that of (4)-(5).
In order to develop the time error estimators, we use the continuous Galerkin method by considering the functions $(\tilde{\mathbf{u}}_\tau, \mathbf{u}_\tau)$ which are affine on each interval $[t_{n-1}, t_n]$, $1 \le n \le N$, and equal to the velocities $(\tilde{\mathbf{u}}^n, \mathbf{u}^n)$ at t_n, $0 \le n \le N$. In this case, we observe that we have the relation $\tilde{\mathbf{u}}_\tau - \mathbf{u}_\tau = \tau_n\Phi_\tau^*$, satisfied on each $[t_{n-1}, t_n]$, $1 \le n \le N$, where Φ_τ^* denotes the discontinuous affine function $\Phi_\tau^*(t) = \frac{\tau_{n-1}}{\tau_n}\Phi^{n-1} + \frac{t-t_{n-1}}{\tau_n}(\Phi^n - \frac{\tau_{n-1}}{\tau_n}\Phi^{n-1})$.
It is also usefull to define, by the same way, the discrete counterpart of the functions

1. see the analysis of [6] for the incremental projection scheme.

$(\tilde{\mathbf{u}}_\tau, \Phi^*_\tau)$ associated to the discrete solutions $(\tilde{\mathbf{u}}^n_h, \Phi^n_h)_{0\le n\le N}$, which we denote by $(\tilde{\mathbf{u}}_{h\tau}, \Phi^*_{h\tau})$. Then, we check that $(\tilde{\mathbf{u}}_\tau, \Phi^*_\tau)$ is solution to

$$\left\{\begin{array}{lll} \partial_t \tilde{\mathbf{u}}_\tau - \nu\Delta\tilde{\mathbf{u}}_\tau + \nabla\Phi^*_\tau = \mathcal{F}^n & \text{in} & \Omega\times]t_{n-1}, t_n], \\ \nabla\cdot\tilde{\mathbf{u}}_\tau - \tau_n\Delta\Phi^*_\tau = 0 & \text{in} & \Omega\times]t_{n-1}, t_n], \\ \tilde{\mathbf{u}}_\tau = 0, \quad \nabla\Phi^*_\tau\cdot\vec{n} = 0 & \text{on} & \Gamma\times]t_{n-1}, t_n], \end{array}\right. \tag{6}$$

where the functional $\mathcal{F}^n = \mathbf{f}^n - \nu\Delta(\tilde{\mathbf{u}}_\tau - \tilde{\mathbf{u}}^n) + \nabla(\Phi^*_\tau - \frac{\tau_{n-1}}{\tau_n}\Phi^{n-1})$. The problem (6) can be viewed as a singularly perturbed system[2] with perturbation parameter τ_n. Then, by substracting (1) from (6), we derive the first family of time error estimators,

$$\tilde{\zeta}_n = \left(\nu\frac{\tau_n}{3}\right)^{\frac{1}{2}} \left|\tilde{\mathbf{u}}^n_h - \tilde{\mathbf{u}}^{n-1}_h\right|_1 \quad \text{and} \quad \zeta_n = \frac{1}{2}|\tau_n\Phi^n_h - \tau_{n-1}\Phi^{n-1}_h|_1.$$

In particular, we denote by $\tilde{S}_n = \|\mathbf{f} - \mathbf{f}^n\|_{L^2(t_{n-1},t_n;X')}$ the error on the data $\mathbf{f}$.
Next, the second family of error estimators related to space discretization, is derived from the difference between the schemes (2)-(3) and (4)-(5). In fact, for $n = 1\dots N$ and for each $K\in\mathcal{T}^n_h$, we define the *local* error estimators

$$\tilde{\eta}_{n,K} = h_K\left\|\mathbf{f}^n_h - \frac{\tilde{\mathbf{u}}^n_h - \tilde{\mathbf{u}}^{n-1}_h}{\tau_n} - \frac{\tau_{n-1}}{\tau_n}\nabla\Phi^{n-1}_h + \nu\Delta\tilde{\mathbf{u}}^n_h\right\|_{0,K} + \frac{\nu}{2}\sum_{E\subset\partial K\cap\Omega} h_E^{\frac{1}{2}}\|[\vec{n}_E.\nabla\tilde{\mathbf{u}}^n_h]\|_{0,E},$$

and $\eta_{n,K} = h_K\|\frac{1}{\tau_n}\nabla.\tilde{\mathbf{u}}^n_h - \Delta\Phi^n_h\|_{0,K} + \frac{1}{2}\sum_{E\subset\partial K\cap\Omega} h_E^{\frac{1}{2}}\|[\nabla\Phi^n_h\cdot\vec{n}_E]\|_{0,E}$.

Here, h_K and h_E denote respectively the diameter of K and the diameter of the side or face $E\subset\partial K$, $[\cdot]$ denote the jump of a given function across E in the direction of its unit normal vector $\vec{n}_E$ external to K and $\mathbf{f}^n_h$ is an approximation of the data $\mathbf{f}^n$. For a simple convenience, we respectively denote by $\tilde{\eta}_n$, η_n and S_n the Hilbertian sum on all the $K\in\mathcal{T}^n_h$ of $\tilde{\eta}_{n,K}$, $\eta_{n,K}$ and $h_K\|\mathbf{f}^n - \mathbf{f}^n_h\|_{0,K}$.

4. Equivalence with the error

Now, we establish how the estimators can bound the error from below and from above for a convenient norm. In this context, we introduce for $m = 1\dots N$, the quantities :

$$\tilde{\zeta}_{m\tau} = \left\{\sum_{n=1}^m \tilde{\zeta}^2_n\right\}^{\frac{1}{2}}, \quad \zeta_{m\tau} = \left\{\sum_{n=1}^m \zeta^2_n\right\}^{\frac{1}{2}}, \quad S_{m\tau} = \left\{\sum_{n=1}^m S^2_n\right\}^{\frac{1}{2}},$$

$$\tilde{\eta}_{mh} = \left\{\sum_{n=1}^m \tau_n\tilde{\eta}^2_n\right\}^{\frac{1}{2}}, \quad \eta_{mh} = \left\{\sum_{n=1}^m \tau^2_n\eta^2_n\right\}^{\frac{1}{2}} \quad \text{and} \quad S_{mh} = \left\{\sum_{n=1}^m \tau_n S^2_n\right\}^{\frac{1}{2}};$$

we also define the global errors in time and space respectively by :

$$E_\tau(t_m) = [\mathbf{u} - \tilde{\mathbf{u}}_\tau](t_m) + \left\{\sum_{n=1}^m \tau_n\int_{t_{n-1}}^{t_n} |p - \Phi^*_\tau|^2_1 dt\right\}^{\frac{1}{2}},$$

2. cf. [10] for similar analysis on the convergence properties in time of the projection scheme and its variants

$$E_h(t_m) = [\tilde{\mathbf{u}}_\tau - \tilde{\mathbf{u}}_{h\tau}](t_m) + \left\{ \sum_{n=1}^{m} \tau_n \int_{t_{n-1}}^{t_n} |\Phi_\tau^* - \Phi_{h\tau}^*|_1^2 dt \right\}^{\frac{1}{2}}.$$

Thus, with the above notations, we can establish the main result of the present study :

Theorem 1 *We assume that Ω is H^2-elliptic and $(\mathbf{u}_0, \mathbf{f}) \in \mathbf{V} \times L^2(0,T;Y) \cap \mathcal{C}^0(0,T;X')$. Then, the following a posteriori error estimations hold*

$$\nu^{-\frac{1}{2}} \|\partial_t(\mathbf{u} - \tilde{\mathbf{u}}_\tau)\|_{L^2(0,t_m;\mathbf{V}')} + E_\tau(t_m) + \nu^{-\frac{1}{2}} \|\partial_t(\tilde{\mathbf{u}}_\tau - \tilde{\mathbf{u}}_{h\tau})\|_{L^2(0,t_m;\mathbf{V}')} + E_h(t_m) \leq \mathbf{I}_r^m, \quad (7)$$

where,

$$\mathbf{I}_r^m = c_2(\tilde{\zeta}_{m\tau}^2 + \zeta_{m\tau}^2)^{\frac{1}{2}} + c_1 \nu^{-\frac{1}{2}} S_{m\tau} + c^\top \nu^{-\frac{1}{2}} \left(\frac{1+\sigma_\tau}{2} (\tilde{\eta}_{mh}^2 + \nu \eta_{mh}^2 + S_{mh}^2) \right)^{\frac{1}{2}}$$

$$+ c_0 [\max_{1 \leq n \leq m} \tau_n] \mathbf{C}_m(\nu, \mathbf{f}, \mathbf{u}_0) + c_0' \left(\frac{1+\sigma_\tau}{2} \|\mathbf{u}_0 - \pi_h \mathbf{u}_0\|_0^2 + \tau_1^2 |\mathbf{u}_0 - \pi_h \mathbf{u}_0|_1^2 \right)^{\frac{1}{2}},$$

with $\mathbf{C}_m(\nu, \mathbf{f}, \mathbf{u}_0) = \left\{ 2\|\mathbf{f}\|_{L^2(0,t_m;Y)}^2 + \nu |\mathbf{u}_0|_1^2 \right\}^{\frac{1}{2}}$.

Then, if the condition $\sup_{1 \leq n \leq N} \sup_{K \in \mathcal{T}_h^n} \frac{h_K^2}{\nu \tau_n} \simeq 1$ *is satisfied, we derive the second estimation*

$$\nu^{-\frac{1}{2}} \|\partial_t(\mathbf{u} - \tilde{\mathbf{u}}_\tau)\|_{L^2(0,t_m;X')} + E_\tau(t_m) + \nu^{-\frac{1}{2}} \|\partial_t(\tilde{\mathbf{u}}_\tau - \tilde{\mathbf{u}}_{h\tau})\|_{L^2(0,t_m;X')} + E_h(t_m) \geq \mathbf{I}_l^m, \quad (8)$$

where, $\mathbf{I}_l^m = c'(\tilde{\zeta}_{m\tau}^2 + \zeta_{m\tau}^2)^{\frac{1}{2}} + c_\perp \nu^{-\frac{1}{2}} \left(\tilde{\eta}_{mh}^2 + \nu \eta_{mh}^2 \right)^{\frac{1}{2}}$.
The constants c_0, c_0', c_1, c_2, and c', are positif and independent of any time or a mesh size, while $c^\top$ and $c_\perp$ depends on the maximal ratio of any element K to the diameter of its largest inscribed ball. In addition, $c_\perp$ depend on the maximum degree of the finite element polynomial functions.

5. Algorithm

In order to adapt the time step and mesh size, we consider a given tolerance Tol, two reals γ_t and γ_h such that $\gamma_t + \gamma_h \leq 1$. A parameter $\delta_1 \in]0,1[$ is used to reduce the step size until the estimator is below Tol. If the error estimator is *much* smaller than the bound Tol, the step size is enlarged by a factor $\delta_2 > 1$. To keep the algorithm robust, we add the parameters $\theta_1, \theta_3 \in]0,1[$, $\theta_2 \in]0, \theta_1[$ and $\theta_4 \in]0, \theta_3[$. Then, with an initial guess τ_1 and mesh $\mathcal{T}_h^0$, the algorithm reads :

$t_0 := 0$ $t_1 := \tau_1$
Resolve (4)-(5) on $\mathcal{T}_h^0$
Compute $\tilde{\zeta}_1$ and ζ_1
while $t_n \leq T$, $(\tilde{\zeta}_{n\tau}^2 + \zeta_{n\tau}^2)^{\frac{1}{2}} \leq \gamma_t$Tol and $\nu^{-\frac{1}{2}} \left(\tilde{\eta}_{nh}^2 + \nu \eta_{nh}^2 \right)^{\frac{1}{2}} \leq \gamma_h$Tol
    $\tau_n := \tau_{n-1}$

$t_n := t_{n-1} + \tau_n$
Resolve (4)-(5)
Compute $\tilde{\zeta}_n$ and ζ_n
while $\max(\tilde{\zeta}_n, \zeta_n) \geq \theta_1 \gamma_t$Tol
$\tau_n := \delta_1 \tau_n$
$t_n := t_{n-1} + \tau_n$
Resolve (4)-(5)
Compute $\tilde{\zeta}_n$ and ζ_n
end while
$\forall K \in \mathcal{T}_h^{n-1}$ Compute $\tilde{\eta}_K$ and η_K, then $\tilde{\eta}_n$ and η_n
while $\tau_n(\nu^{-\frac{1}{2}}\tilde{\eta}_n + \tau_n \eta_n) \geq \theta_3 \gamma_h$Tol
Refine $\mathcal{T}_h^{n-1}$ producing $\mathcal{T}_h^n$
Resolve (4)-(5)
$\forall K \in \mathcal{T}_h^{n}$ Compute $\tilde{\eta}_K$ and η_K, then $\tilde{\eta}_n$ and η_n
end while
if $\tau_n(\nu^{-\frac{1}{2}}\tilde{\eta}_n + \tau_n \eta_n) \leq \theta_4 \gamma_h$Tol
Adapt mesh $\mathcal{T}_h^n$ for coarsening
end if
Compute $\tilde{\eta}_{nh}$ and η_{nh}
if $\max(\tilde{\zeta}_n, \zeta_n) \leq \theta_2 \gamma_t$Tol
$\tau_n := \delta_2 \tau_n$
end if
Compute $\tilde{\zeta}_{n\tau}$ and $\zeta_{n\tau}$
end while

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