Iterative behaviour of quasi-palindromic neural networks with memory

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ABSTRACT. In this paper, we study dynamics of neural network with memory where the updating consider a longer history of each site and the set of interaction matrices is quasi-palindromic. For parallel and sequential iteration, we define Lyapunov functionals which permits us to characterize the periods behaviour and explicitely bounds the transient lengths of quasi-palindromic neural networks. For these networks, due to the quasi-palindromy, the dynamic is robust with respect to a class of small changes of the interactions matrices. This property is important in many applications of neural networks such as association, optimization and pattern recognition.

RÉSUMÉ. Nous étudions la dynamique des réseaux de neurones avec mémoire où la mise à jour considère une suite d'états de chaque site et où l'ensemble des matrices d'interaction est quasi-palindromique. Nous définissons pour l'itération parallèle et l'itération série des fonctionnelles de Lyapunov qui nous permettent de caractériser les périodes et de borner les longueurs des transitoires des réseaux de neurones étudiés. La dynamique de ces réseaux, grâce à la quasi-palindromie, est assez robuste face à certaines perturbations des interactions entre neurones. Cette propriété est importante dans de nombreuses applications des réseaux de neurones telles les mémoires associatives, l'optimisation et la reconnaissance.

KEYWORDS : Neural network with memory, Transient length, Period, Lyapunov functional, Quasi-palindromic.

MOTS-CLÉS : Réseau de neurones à mémoire, Longueur de transitoire, Période, Fonctionnelle de Lyapunov, Quasi-palindromique.

1. Introduction

Neural network were introduced by Mc Culloch and Pitts [15] and they are seen as merely one example of a class of complex system [6]. A neural network is a statistical construct adept at inferring functions and thereby mapping inputs to corresponding outputs. Neural networks are more than mere statistical techniques; a neural network meeting certain simple preconditions becomes Turing universal. Because of Turing universality, the neural network becomes, in a computational theoretic sense, an alternative model to Turing machines and their Von Neumann descendants. Nneural networks have been studied extensively as tools for solving various problems such as classification, speech recognition, image processing [7]. These applications rely on the stability of existing fixed points and cycles of iteration graphs of neural networks.

Neural network is usually implemented by using electronic components or is simulated in software on a digital computer. One way in which the collective properties of a neural network may be used to implement a computational task is by way of the concept of *energy minimization*. The Hopfield network has attracted a great deal of attention in the literature as a *content-addressable memory* [12].

Caianiello [1] has suggested that the dynamic behaviour of a neuron in a neural network with *k-memory* can be modeled by the following recurrence equation:

$$x_i(t) = \mathbf{1}\left(\sum_{j=1}^{n}\sum_{s=1}^{k} a_{ij}(s)\, x_j(t-s) - b_i\right), \ \ t \geq k \tag{1}$$

where

– i is the index of a neuron, $i = 1, ..., n$.

– $x_i(t) \in \{0, 1\}$ is a variable representing the state of the neuron i at time t.

– k is the memory length, i.e., the state of a neuron i at time t depends on the states $x_j(t-1), ..., x_j(t-k)$ assumed by all the neurons ($j = 1, ..., n$) at the previous steps $t-1, ..., t-k$ ($k \geq 1$).

– $a_{ij}(s)$ ($1 \leq i, j \leq n$ and $1 \leq s \leq k$) are real numbers called the weighting coefficients. More precisely, $a_{ij}(s)$ represents the influence of the state of the neuron j at time $t-s$ on the state assumed by the neuron i at time t.

– b_i is a real number called the threshold.

– $\mathbf{1}$ is the Heaviside function: $\mathbf{1}(u) = 0$ if $u < 0$, and $\mathbf{1}(u) = 1$ if $u \geq 0$

For evolution Eq.(1), Goles [8] show that if the class of interaction matrices is palindromic the periods T divide $k+1$. Tchuente [18] generalized the preceding result by showing that the parallel iteration of a network of automata N can be sequentially simulated by another network N' whose local transition functions are the same as those of N. In [17], the transient length of the trajectory generated by evolution Eq.(1), when the class of interaction matrices is palindromic, was bounded. By implementing a binary Borrow-Save counter, Ndoundam and Tchuente [16] exhibit a Caianiello automata network of size $2n+2$ and memory length k which describes a cycle of length $k2^{nk}$. Evolution Eq.(1) corresponds to a parallel updating of neurons. The sequential updating of neural network with memory was introduced in [17] where it was shown that if the class of interaction matrices is palindromic and the diagonals of matrices are equals, with non-negative elements, then the periods T of the neural network iterated sequentially with k-memory satisfy $T|k$. The dynamics generated by Eq.(1) have been studied for some particular one-dimensional systems: when $n = 1$, one obtains a single neuron (proposed by Caianiello and De Luca [2]) with memory that does not interact with other neurons. When $k = 1$, one obtains a Mc Culloch and Pitts neural network [15]. It has been proved ([9, 10]) that Mc Culloch and Pitts neural networks defined by symmetric matrices admit Lyapunov functional. Later, this result was proved for quasi-symmetric weights generalizing the symmetry property [10, 14].

In complex systems theory, the prevailing view is that a system's regime of dynamical behaviour largely determines the system's capacity to process information, perform computations, and generally exhibit sophisticated behaviour and self-organization of complex structure [11]. Only systems on an extended transient trajectory are capable of evolving in a way that encodes significant information. Moreover, strong parallels can be drawn between the behaviour of dynamical systems and the properties of computational systems [5]. Our motivation for studying the dynamics generated by iteration of neural network with memory is to examine the transient and cycle lengths that can be generated by it in the special case where non-trivial regularities on coupling coefficients are satisfied. Our approach consists to define appropriate Lyapunov functional [13].

It is well known that the collective dynamics of neural networks essentially relies on connectivity properties of the systems; Mc Culloch and Pitts neural networks with symmetric connection have convergent dynamics [3, 13]. Whereas, the dynamics of such asymmetric neural networks can be diverse and can demonstrate convergence, oscillation or chaotic behavior [3]. Stable and convergent dynamics is an essential property of neural networks and is important in many applications of neural networks such as association, optimization and pattern recognition [3, 13]. Stability means that the concerned neural network possesses some attractive equilibrium points for every constant input and every interaction matrices of certain type. We study in this paper neural networks with a quasi-palindromic set of interaction matrices and show that, these networks always converge to equilibrium points of length related to the length of the memory.

The remainder of the paper is organized as follows: in Section 2, some definitions and notations are given. In Section 3, we characterize the periods and bound the transient lengths of parallel iteration of neural networks with memory of which the set of interaction matrices is quasi-palindromic. The same study is made in Section 4 for sequential iteration of neural networks with memory of which the set of interaction matrices is quasi-palindromic. Concluding remarks are stated in Section 5.

2. Definitions and notations

A neural network N iterated with a memory of length k is defined by $N = (I, A(1), ..., A(k), b)$, where $I = \{1, ..., n\}$ is the set of neurons indexes, $A(1), ..., A(k)$ are matrices of interactions and $b = (b_i : i \in \{1, ..., n\})$ is the threshold vector. Let $\{x(t) \in \{0,1\}^n : t \geq 0\}$ be the trajectory starting from $x(0)$,..., $x(k-1)$; since $\{0,1\}^n$ is finite, this trajectory must sooner or later encounter a state that occurred previously - it has entered an *attractor cycle*. The trajectory leading to the attractor is a *transient*. The period (T) of the attractor is the number of states in its cycle, which may be just one - a fixed point. If $(x(0), ..., x(T-1))$ is a T-cycle, then the T-cycle at site i is denoted $X_i = (x_i(0), ..., x_i(T-1))$. The period of X_i is denoted $\gamma(X_i)$, by definition $\gamma(X_i) | T$. The transient length of the trajectory is noted $\tau(x(0), ..., x(k-1))$. The transient length of the neural network is defines as the greatest of transient lengths of trajectories, that is:

$$\tau(A(1), ..., A(k), b) = max\{\tau(x(0), ..., x(k-1)) : x(t) \in \{0,1\}^n, \ 0 \leq t \leq k-1\}$$

The updates of the state values of each neuron depends on the type of iteration associated to the model. The parallel iteration consists of updating the value of all the neurons at the same time. The sequential iteration consists of one by one updating the neurons in a pre-established periodic order $(i_1, i_2, ..., i_n)$, where $I = \{i_1, i_2, ..., i_n\}$.

Definition 1 *A set of interaction matrices $(A(1), ..., A(k))$ is quasi-palimdromic if $\forall i, j \in I, \exists \lambda_i, \lambda_j > 0$ such that $\forall s = 1, ..., k$, $\lambda_i a_{ij}(k - s + 1) = \lambda_j a_{ji}(s)$, i.e., if Λ is the diagonal matrix of λ_i ($\Lambda_{ii} = \lambda_i$, $\Lambda_{ij} = 0\ \forall i \neq j$), $\Lambda.A(k - s + 1) = \Lambda.A(s)^t$. By extension a neural network $N = (I, A(1), ..., A(k), b)$ is quasi-palindromic if $(A(1), ..., A(k))$ is quasi-palimdromic.*

Remark that when $\lambda_i = 1, \forall i \in I$, we get a palindromic set of interaction matrices. Hence quasi-palindromy can be interpreted as a direct generalization of palindromy. Note also that if $k = 1$, quasi-palindromy corresponds to quasi-symmetry [4, 10, 14].

The notion of Lyapunov functional has been introduced in the study of neural networks in order to study the dynamics of symmetric and quasi-symmetric Mc Culloch and Pitts neural networks [10, 13].

Definition 2 *[10] For a dynamics $x(t + 1) = F(x(t), x(t - 1), ...)$ a real functionnal $E(x(t))$ is called a Lyapunov functional if it is decreasing: $E(x(t + 1)) \leq E(x(t))$ for any $t \geq 1$.*

From this definition, it is direct to show that, if $(x(0), ..., x(T - 1))$ is a T-cycle, then necessarily the functional is constant on it, i.e. $E(x(0)) = ... = E(x(T - 1))$. The existence (or non-existence) of Lyapunov functional driving the network dynamics is extremly sensitive to small perturbations on the weights, i.e., small alterations to the interaction matrices may change completely the dynamic behaviour of the network [10]. However the Lyapunov functional for quasi-palindromic neural networks are very robust in the sense that any neural network with interaction matrices derived from the initial ones by local operations which preserve the quasi-palindromy also accepts the same kind of Lyapunov functional, this time acting on the new interaction matrices and threshold vector.

Let $N = (I, A(1), ..., A(k), b)$ a neural network with a memory of length k. Let us note:

$$e_i = min \left\{ \left| \sum_{j=1}^{n} \sum_{s=1}^{k} a_{ij}(s)\, u_j(s) - b_i \right| : u(s) \in \{0,1\}^n,\ s = 1, ..., k \right\} \tag{2}$$

We can assume that:

$$\sum_{j=1}^{n} \sum_{s=1}^{k} a_{ij}(s)\, u_j \neq b_i,\ \ \forall i \in I,\ \forall u = (u_1, ..., u_n) \in \{0,1\}^n \tag{3}$$

In fact, if $\sum_{j=1}^{n} \sum_{s=1}^{k} a_{ij}(s)\, u_j = b_i$, it suffices to make small change in the hyperplane coefficients, or the threshold, in oder to avoid this situation without modifying the dynamics of the network [10]. We will use the following notations: $diag(A(s)) = (a_{ii}(s) : i \in I)$, $\|A(s)\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}(s)|$ for $s = 1, ..., k$, $\|u\| = \sum_{i=1}^{n} |u_i|$ for any vector $u \in \mathbf{R}^n$ and $\bar{1} = (1, ..., 1)^t$.

3. Parallel iteration

Let us consider the parallel iteration of a finite neural network with memory $N = (I, A(1), ..., A(s), b)$ give by Eq.(1). Let $\lambda = (\lambda_i : i \in I)$ $(\forall i \in I,\ \lambda_i > 0)$ and let $\{x(t) : t \geq 0\}$ be a trajectory of the parallel iteration, we define the following functional for $t \geq k$:

$$E_{par}(x(t)) = -\sum_{i=1}^{n}\left(\sum_{s=0}^{k-1} x_i(t-s)\left(\sum_{j=1}^{n}\sum_{s'=1}^{k-s}\lambda_i a_{ij}(s')\, x_j(t-s-s')\right) - \lambda_i b_i \sum_{s=0}^{k} x_i(t-s)\right) \tag{4}$$

Proposition 1 *The functional $E_{par}(x(t))$ is a strictly decreasing Lyapunov functional for the parallel iteration of quasi-palindromic neural network with memory.*

Using the preceding functional, we will characterize the periods of neural networks.

Theorem 1 *The periods T of parallel iteration of quasi-palindromic neural networks with memory satisfies $T|k+1$.*

We will now bound the transient length of neural networks. Denotes by $\bar{X}$ the set of all initial conditions which do not belong to a period of length $k+1$:

$$\bar{X} = \{x(0) \in \{0,1\}^n \text{ such that } x(0) \neq x(k+1)\}$$

If $\bar{X} \neq \emptyset$ define:

$$e = min\left\{-\left(E(x(k+1)) - E(x(k))\right) : x(0) \in \bar{X}\right\} \tag{5}$$

We note $e = 0$ if $\bar{X} = 0$.

Theorem 2 *The transient length $\tau_{par}(A(1), ..., A(k), \lambda, b)$ of parallel iteration of quasi-palindromic neural network with memory is bounded by:*

$$\begin{aligned} \tau_{par}(A(1), ..., A(k), \lambda, b) \leq\ & \tfrac{1}{4e}\left((k+2)\left\|\left(2b - \sum_{s=1}^{k} A(s)\,\bar{1}\right).\Lambda\right\|\right. \\ & \left. + k\sum_{s=1}^{k}\|\Lambda.A(s)\| - 2k\sum_{i=1}^{n}\lambda_i e_i\right) \quad if\ \ e > 0 \\ \tau_{par}(A(1), ..., A(k), \lambda, b) =\ & 0 \qquad if\ \ e = 0 \end{aligned} \tag{6}$$

Remark 1 Theorem 1 and Theorem 2 are generalization of some results established in [17] for palindromic neural networks with memory.

Remark 2 For $k = 1$, one get a Mc Culloch and Pitts neural network for which the interaction matrix is quasi-symmetric and our results can be seen as a generalization of those obtained in [10].

4. Sequential iteration

The sequential updating of the neural network with memory is written [17]:

$$x_i(t) = \mathbf{1}\left(\sum_{j=1}^{i-1}\sum_{s=1}^{k} a_{ij}(s)\, x_j(t+1-s) + \sum_{j=i}^{n}\sum_{s=1}^{k} a_{ij}(s)\, x_j(t-s) - b_i\right) \tag{7}$$

Let $\lambda = (\lambda_i : i \in I)$ $(\forall i \in I,\ \lambda_i > 0)$. Let $\{x(t) : t \geq 0\}$ be a trajectory of the sequential iteration, we define the following functional for $t \geq k$ $(k > 1)$:

$$\begin{aligned} E_{seq}(x(t)) = \; & -\sum_{i=1}^{n}\left(\sum_{s=0}^{k-1} x_i(t-s)\sum_{j<i}\sum_{s'=1}^{k-s}\lambda_i a_{ij}(s')\, x_j(t-s-s'+1)\right. \\ & \left. + \sum_{s=1}^{k-1} x_i(t+1-s)\sum_{j\geq i}\sum_{s'=1}^{k-s}\lambda_i a_{ij}(s')\, x_j(t-s-s'+1) - \lambda_i b_i \sum_{s=0}^{k-1} x_i(t-s)\right) \end{aligned} \tag{8}$$

Proposition 2 *If the class of interaction matrices $(A(s) : s = 1, ..., k)$ satisfies:*

– $diag(A(s)) = diag(A(s+1)) \quad \forall s = 1, ..., k-1$

– $\forall i \in I,\ \ a_{ii}(k) \geq 0$

then the functional $E_{seq}(x(t))$ is a strictly decreasing Lyapunov functional for the sequential iteration of quasi-palindromic neural network with memory.

We will now characterize the periods of trajectory generated by neural networks.

Theorem 3 *If the class of interaction matrices $(A(s) : s = 1, ..., k)$ satisfies:*

– $diag(A(s)) = diag(A(s+1)) \quad \forall s = 1, ..., k-1$

– $\forall i \in I,\ \ a_{ii}(k) \geq 0$

then the periods T of sequential iteration of quasi-palindromic neural networks with memory satisfies $T|k$.

To study the transient phase, we will work with another Lyapunov functional derived from $E_{seq}(x(t))$. Define:

$$\begin{aligned} E^*_{seq}(x(t)) = \; & -\sum_{i=1}^{n}\left(\sum_{s=0}^{k-1}(2x_i(t-s)-1)\sum_{j<i}\sum_{s'=1}^{k-s}\lambda_i a_{ij}(s')(2x_j(t-s-s'+1)-1)\right. \\ & \left. + \sum_{s=1}^{k-1}(2x_i(t+1-s)-1)\sum_{j\geq i}\sum_{s'=1}^{k-s}\lambda_i a_{ij}(s')(2x_j(t-s-s'+1)-1)\right) \\ & + \sum_{i=1}^{n}\left(\left(2\lambda_i b_i - \sum_{j=1}^{n}\sum_{s=1}^{k}\lambda_i a_{ij}(s)\right)\sum_{s=0}^{k-1}(2x_i(t-s)-1)\right) \\ & + \sum_{i=1}^{n}\sum_{s=1}^{k}\lambda_i a_{ii}(s)(2x_i(t-s+1)-1) \end{aligned} \tag{9}$$

Proposition 3 *If the class of interaction matrices $(A(s) : s = 1, ..., k)$ satisfies:*

– $diag(A(s)) = diag(A(s+1)) \quad \forall s = 1, ..., k-1$

– $\forall i \in I,\ \ a_{ii}(k) \geq 0$

then the functional $E^*_{seq}(x(t))$ is a strictly decreasing Lyapunov functional for the sequential iteration of quasi-palindromic neural network with memory.

Now, we assume that the set of interaction matrices $(A(s) : s = 1, ..., k)$ satisfiy the following conditions:

– The diagonals of interaction matrices are equivalent:

$$diag\left(A\left(s\right)\right) = diag\left(A\left(s+1\right)\right) \quad for \;\; s = 1, ..., k-1 \tag{10}$$

– The elements of these diagonals are non-negative:

$$\forall i \in I, \;\; a_{ii}\left(k\right) \geq 0 \tag{11}$$

Let us denotes by $\bar{X}'$ the set of all initial conditions which do not belong to a period of length k:

$$\bar{X}' = \left\{x\left(0\right) \in \{0,1\}^n \text{ such that } x\left(0\right) \neq x\left(k\right)\right\}$$

Recall that $\bar{X}'$ is empty iff the transient length of the neural network is null. If $\bar{X}' \neq \emptyset$ define:

$$e' = min\left\{-\left(E\left(x\left(k\right)\right) - E\left(x\left(k-1\right)\right)\right) : x\left(0\right) \in \bar{X}'\right\} \tag{12}$$

We note $e' = 0$ if $\bar{X}' = 0$.

Proposition 4 *Let $\{x\left(t\right) : t \geq 0\}$ be a trajectory for a quasi-palindromic neural network with k-memory iterated sequentially; $E^*_{seq}\left(x\left(t\right)\right)$ is bounded by:*

$$\begin{aligned} E^*_{seq}\left(x\left(t\right)\right) \geq \;\; & -\sum_{i=1}^{n}\sum_{j<i}\sum_{s=1}^{k}(k-s+1)\lambda_i\left|a_{ij}\left(s\right)\right| - \sum_{i=1}^{n}\sum_{j\geq i}\sum_{s=1}^{k}(k-s)\lambda_i\left|a_{ij}\left(s\right)\right| \\ & -k\left\|\left(2b - \sum_{s=1}^{k}A\left(s\right).\bar{1}\right).\Lambda\right\| - k\left\|diag(A(k)).\Lambda\right\| \end{aligned} \tag{13}$$

and

$$\begin{aligned} E^*_{seq}\left(x\left(t\right)\right) \leq \;\; & \sum_{i=1}^{n}\sum_{j<i}\sum_{s=1}^{k}(s-1)\lambda_i\left|a_{ij}(s)\right| + \sum_{i=1}^{n}\sum_{j\geq i}\sum_{s=1}^{k}s\lambda_i\left|a_{ij}(s)\right| \\ & -2k\sum_{i=1}^{n}\lambda_i e_i + k\left\|diag(A(k)).\Lambda\right\| \end{aligned} \tag{14}$$

We will now bound the transient lengths of the sequential iteration of quasi-palindromic neural networks with memory.

Theorem 4 *If the class of interaction matrices $\left(A\left(s\right) : s = 1, ..., k\right)$ satisfies:*

– $diag\left(A\left(s\right)\right) = diag\left(A\left(s+1\right)\right) \quad \forall s = 1, ..., k-1$

– $\forall i \in I, \;\; a_{ii}\left(k\right) \geq 0$

then the transient length $\tau_{seq}\left(A\left(1\right), ..., A\left(k\right), \lambda, b\right)$ of sequential iteration of neural network with memory is bounded by:

$$\begin{aligned} \tau_{seq}\left(A\left(1\right), ..., A\left(k\right), \lambda, b\right) \leq \;\; & \tfrac{k}{4e'}\left(\left\|\left(2b - \sum_{s=1}^{k}A\left(s\right).\bar{1}\right).\Lambda\right\| + \sum_{s=1}^{k}\left\|\Lambda.A\left(s\right)\right\|\right. \\ & \left. +2\left\|diag(A(k)).\Lambda\right\| - 2\sum_{i=1}^{n}\lambda_i e_i\right) \;\; if \;\; e' > 0 \end{aligned} \tag{15}$$

$$\tau_{seq}\left(A\left(1\right), ..., A\left(k\right), \lambda, b\right) = \;\; 0 \quad\quad if \;\; e' = 0$$

Remark 3 When $\lambda_i = 1, \forall i \in I$, it was shown in [17] for sequential iteration that, the conditions given by Eqs.(10), (11) are necessary so that the periods divide k.

5. Conclusion

We study neural networks of Caianiello under some assumptions on interaction matrices. For parallel iteration, using Lyapunov functional, we characterize the periods and bounds explicitely the transient lengths of quasi-palindromic neural networks. These results generalize those obtained for palindromic Caianiello neural networks, for symmetric and quasi-symmetric Mc Culloch and Pitts neural networks. For sequential iteration we characterize the periods and bounds explicitely the transient lengths of quasi-palindromic neural networks of which the diagonals of interaction matrices are equals and their elements are non-negative.

For future work, it would be interesting to study the stability of the quasi-symmetric neural networks with memory to perturbations in order to determine the conditions in which they can be used for applications such as association, optimization and pattern recognition.

Acknowledgements

This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre of Theoretical Physics (ICTP), Trieste, Italy. Finacial support from the Swedish International Development Cooperation Agency (SIDA) is acknowledged.

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