The incorporation of topological derivative into level set methods for cavity identification

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ABSTRACT. The present paper is concerned with the identification of cavities of different conductivities included in a two-dimensional domain by measurments of voltage and curents at the boundary. We refomulate the given identification problem as a shape optimization problem. The representation of the shape of the boundary and its evolution during an iterative reconstruction process is achieved by the level set methods. we investgated the use of topological derivative in combination with the level set methods for shape optimization. The topological derivative indicates the appropriate location of cavities. Finally the shape is corrected by level set methods.

RÉSUMÉ. Le présent papier porte sur l'identification des cavités inclues dans un domaine bi-dimensionnel de conductivité non constante par des mesures de la tension et du courant électrique sur la frontière. Nous le reformulons comme un problème d'optimisation de forme. La représentation de la géométrie et de son évolution au cours d'un processus itératif de reconstruction est réalisée par la méthode "level set". Nous proposons la combinaison du gradient topologique avec la méthode level set pour l'optimisation de forme, le gradient topologique nous indique l'emplacement des cavités. Enfin, la forme est corrigée par la méthode "level set".

KEYWORDS : Geometric inverse problem, level set methods, topological gradient

MOTS-CLÉS : Problème inverse géométrique, méthode level set, gradient topologique

1. Introduction and problem statement

Suppose that Ω is an electrically conducting body with boundary $\partial\Omega = \Gamma$. Let μ denote the conductivity of the medium, u the potential and g the known boundary sources. The governing equations are

$$\left\{ \begin{array}{lcll} -div(\mu\nabla u) & = & 0 & \text{in} \quad \Omega \\ \dfrac{\partial u}{\partial n} & = & g & \text{on} \quad \Gamma \end{array} \right. \tag{1}$$

where n is the unit outer normal direction.

We assume that Ω can be decomposed into two disjoint subdomains Ω^+ and Ω^- and that the value of the conductivity is known in each of them.

Let Γ be the region of observation. The problem consists in determining the interface Σ between Ω^+ and Ω^- from measurements u^* of the potential function u on Γ. There are however, numerous other applications leading to similar formulations.

The conductivity μ is supposed to be piecewise constant and is defined by

$$\mu(x) = \left\{ \begin{array}{ll} \mu^+ & x \in \Omega^+ \\ \mu^- & x \in \Omega^- \end{array} \right.$$

We assume that Ω^- is the finite union of simply connected open sets in Ω.

Their boundary Σ represents the interface between the two open domains Ω^+ and Ω^-, and it is assumed to be the union of closed C^2 curves. We further assume that the interface Σ is strictly contained in $\Omega = \Omega^+ \cup \Omega^-$.
If μ^- is ∞, then Ω^- is a perfect conductor. In the case Ω^- consists only of one connected component, the boundary value problem (1) reduces to

$$\left\{ \begin{array}{lcll} -\Delta u & = & 0 & \text{in} \quad \Omega^+ \\ u & = & 0 & \text{on} \quad \Sigma \\ \dfrac{\partial u}{\partial n} & = & g & \text{on} \quad \Gamma \end{array} \right. \tag{2}$$

The problem under consideration is a special case of the general conductivity reconstruction problem and is severely ill-posed. We emphasize that we focus in the present paper on exact measurements and do not consider noisy data.

The geometric inverse problem can be formulated as minimization of least-square mis-fit function over a class of unknown interface. The shape optimization involves minimizing cetain cost function J over a class of admissible boundary shapes.

In general we consider the minimization of the form

$$\inf_{\Sigma \in \Sigma_{ad}} J(\Sigma),$$

where Σ_{ad} is the admissible class of interfaces. Usually interface identification problems are ill-posed.

1.1. Shape derivatives

Like the Gâteaux and Fréchet derivatives in a analytic function framework, the shape derivative is a fundamental tool for geometric inverse problems, since it allows to characterize extrema and yields directions of steepest descent. For a comprehensive introduction to shape derivatives we refer to [20]. The intention of this section is to give a very brief introduction into shape derivatives.
In the following, we suppose without restriction of generality that Σ is piecewise C^2.

Given a smooth vector field $V(t,x)$, it gives rise to a family of transformations $T_t(X)$ via the differential equation

$$\left\{\begin{array}{lcl} \dot{x}(t) & = & V(t,x(t)) \\ x(0) & = & X \end{array}\right. \quad (3)$$

where every point $X \in \Omega$ is mapped by $T_t(X)$ to the solution $x(t,X)$ of (3) at time t, i.e. $T_t(X)(x) = x(t,X)$. This difines the perturbed domain by

$$\Omega_t = T_t(\Omega)$$

This parameterization of domains was first studied by [12] and enable us to view geometric objects in the usual analytic function contex.
Let v_t be a function defined on the domain Ω_t (e.g. , $v_t = u(\Omega_t)$, where $u(\Omega_t)$ is the solution to (2) with Ω replaced by Ω_t). Then we can consider

$$v' = \frac{\partial v_t}{\partial t}|_{t=0} \qquad \text{shape derivative}$$

$$v^{\bullet} = \frac{\partial v^t}{\partial t}|_{t=0} := \frac{\partial v_t \circ T_t}{\partial t}|_{t=0} \quad \text{material derivative}$$

It is well known that there is a relation between the material derivative $v^{\bullet}$ and the shape derivative v' (see[20]), namely:

$$v' = v^{\bullet} - \nabla v.V.$$

Before we start to calculate the shape derivative for our problem we note that in our problem class of cavity identification we want that the boundary $\partial\Omega$ remains unchanged within the family of transformations T_t. Hence, we may restrict our attention to velocity fields satisfying $V.n = 0$ on $\partial\Omega \setminus \Sigma$, where n denotes the normal vector to the boundary $\partial\Omega \setminus \Sigma$. Furthermore, it suffices to consider only transformations T_t that change the position of Σ but do not "rotate" Σ, which results into vector fields V that have zero tangential part along Σ, i.e.

$$V|_{\Sigma} = (V.n)n = V_n n.$$

2. Topological derivatives

Alternative approaches to the solution of shape reconstruction problems have been considered recently, such as the topological sensitivity analysis was introduced by Schumacher [17], Sokolowski and Zochowski[19]. The given approach is based on the analysis of the topological sensitivity. It provides an asymptotic expansion of a cost function with respect to a small topological perturbation of the domain. To present the basic idea, we consider a domain of $\mathbb{R}^d$, d = 2, 3 and $J(\Omega) = J(\Omega, u_\Omega)$ a cost function to be minimized, where u_Ω is the solution to a given PDE problem defined in Ω. For $\epsilon > 0$, let $\Omega_\epsilon = \Omega \setminus \overline{(x_0 + \epsilon\omega)}$ be the domain obtained by removing a small part $\overline{(x_0 + \epsilon\omega)}$ from Ω, where x_0 and $\omega \subset \mathbb{R}^d$ is a fixed bounded domain containing the origin. Then, an asymptotic expansion of the function J is obtained in the following form :

$$J(\Omega_\epsilon) = J(\Omega) + f(\epsilon)g(x_0) + o(f(\epsilon))$$

$$f(\epsilon) > 0 \ \ \forall \epsilon > 0, \ \ \lim_{\epsilon \to 0} f(\epsilon) = 0$$

The function g is called the topological sensitivity or topological gradient. $g(x_0)$ provides an information for creating a small hole located at x_0: if $g(x_0) < 0$, then $J(\Omega_\epsilon) < J(\Omega)$ for small ϵ.

It can be used as a descent direction of the domain optimization process. Obviously, if we want to minimize J, the "best" place to create an infinitesimal hole is there where $g(x)$ is the most negative.

3. Level Set Methods

Another alternative approaches to the solution of shape reconstruction problems have been considered recently, the level set method (cf [16]) can be applied , which we therefore consider in this paper. The level set approach was introduced by Osher and Sethian [14] for evolving geometries. The main idea is to represent an evolving front as the zero level set of a continuous function Φ,

$$\Sigma_t = \{x \in \mathbb{R}^n / \Phi(x,t) = 0\}$$

A weak formulation of geometric motion with normal speed V_n is given by the Hamilton-Jacobi equation

$$\left\{ \begin{array}{lcll} \dfrac{\partial \Phi}{\partial t} + V_n |\nabla \Phi| & = & 0 & \text{on} \quad \mathbb{R}^n \times \mathbb{R}_+, \\ \Phi(x,0) & = & \Phi_0(x) & \text{on} \quad \mathbb{R}^n \end{array} \right. \tag{4}$$

in the sense that a viscosity solution for this Hamilton-Jacobi equation (4) has to be computed. We refer to the monograph by Lions [11] and the paper by Crandall et al. [7] for details on the notion of viscosity solution. Due to the implicit representation on an Eulerian grid, the level set approach does not introduce any a priori assumptions on the geometry and therefore is receiving growing attention in the context of geometric inverse problems (cf, e.g., [3, 4, 5, 6, 9, 15, 16]) and shape optimization (cf, e.g., [1, 18]).
The basic idea of level set methods for inverse and optimization problems (cf [3, 9, 13, 15, 16]) is to choose the velocity in such a way that a decrease of the objective functional is achieved, which resembles the classical speed method in shape optimization (cf [12, 20]), but the weak formulation via the level set method allows for more general evolution and in particular for topological changes such as splitting or merging of domains.
The numerical solution may not be unique. There can be several local and global minima. Starting with an initial guess that contains all possible expected shapes, the algorithm typically finds an envelope of all shapes representing the minima (cf, e.g.[9]).

4. Shape and Topological gradient calculs

Considering the so-called Kohn-Vogelius criterion as the cost function [10]. The idea is to minimize the misfit between the solutions of two forward problems. Since the boundary conditions u^* and g are over specified, one can define for any Σ the following auxillary problems:

• Dirichlet problem

$$\begin{cases} -\Delta u_D &= 0 \quad \text{in} \quad \Omega^+ \\ u_D &= 0 \quad \text{on} \quad \Sigma \\ u_D &= u^* \quad \text{on} \quad \Gamma \end{cases} \tag{5}$$

• Neumann problem

$$\begin{cases} -\Delta u_N &= 0 \quad \text{in} \quad \Omega^+ \\ u_N &= 0 \quad \text{on} \quad \Sigma \\ \dfrac{\partial u_N}{\partial n} &= g \quad \text{on} \quad \Gamma \end{cases} \tag{6}$$

As mentioned before, the idea behind the Kohn-Vogelius criterion is to minimize the misfit between the solutions of the Dirichlet and Neumann problems. Therefore, the cost function for this inverse problem is given by the following functional

$$J(\Sigma) = \frac{1}{2}||u_D(\Sigma) - u_N(\Sigma)||^2_{L^2(\Omega^+)}$$

Observe that the solutions of both Dirichlet and Neumann problems are the same in all the domain only when $\Sigma = \Sigma^*$. Therefore, if we get the domain in which the cost functional vanishes, then we find the set of unknown cavity Σ^*, which is the solution of our inverse problem.

The shape derivative of the objective functional is J [20]

$$J'(\Sigma, V) = \lim_{t\to 0} \frac{J(\Sigma_t) - J(\Sigma)}{t} = -\int_\Sigma (\nabla u_D \nabla w_D + \nabla u_N \nabla w_N) V_n ds,$$

where the adjoint problems are:

$$\begin{cases} -\Delta w_D &= u_D - u_N \quad \text{in} \quad \Omega^+ \\ w_D &= 0 \quad \text{on} \quad \Sigma \\ w_D &= 0 \quad \text{on} \quad \Gamma \end{cases} \tag{7}$$

$$\begin{cases} -\Delta w_N &= -(u_D - u_N) \quad \text{in} \quad \Omega^+ \\ w_N &= 0 \quad \text{on} \quad \Sigma \\ \dfrac{\partial w_N}{\partial n} &= 0 \quad \text{on} \quad \Gamma \end{cases} \tag{8}$$

We consider the velocity on Σ_t as :

$$V_n = \nabla u_D \nabla w_D + \nabla u_N \nabla w_N$$

To compute the corresponding topological gradient, we need to solve numerically u_D and u_N the two direct problems and w_D and w_N the two adjoint problems on the safe domain Ω: The topological gradient for the case of Dirichlet boundary conditions on the boundary of the ball as in [8, 2]

$$g(x_0) = u_D(x_0) w_D(x_0) + \nabla u_N(x_0) \nabla w_N(x_0)$$

5. Numerical algorithm

Phase I: Initialization

• solve the direct problems (5) (6) for u_D, u_N and their associated adjoint problems (7) (8) whose solutions are w_D and w_N in Ω,
• compute the topological sensitivity g.
• set $\Omega_0 = \Omega \setminus \{x \in \Omega, g(x) < \tau\}$, where the constant τ is chosen by the user.

Phase II

To solve the Hamilton-Jacobi equation (4) numerically, we proceed as follow:
1- The initial level set function Φ_0 which corresponds to the initial form provided by Phase I :

$$\Sigma_0 = \{x \in \mathbb{R}^2; \Phi_0(x) = 0\}$$

2- For $k \geq 0$ and until the algorithm converges.
• Solve for u_D^k, u_N^k solutions of the direct problems (5) (6) and w_D^k, w_N^k solutions of the adjoint problems (7) (8) posed in Ω_k.
• Compute $J(\Sigma_k)$.
• Evaluate the normal velocity V_k on Σ_k.
• Extend the velocity on all Ω.
• Deformation of $\Sigma_k = \{x \in \mathbb{R}^2; \Phi^k(x) = 0\}$ by la solving the Hamilton-Jacobi equation, the new geometrie Σ_{k+1} is donne by la level set fonction Φ^{k+1} solution of the equation:

$$\frac{\Phi^{k+1} - \Phi^k}{\Delta t_k} + V_k|\nabla\Phi^k| = 0$$

When starting from intial function $\Phi^k(x)$ with velocity V_k, $|\nabla\Phi^k|$ evaluated with WENO (weighted essentially non-oscillatory) and Δt_k is chosen as:

$$\Delta t_k = \frac{h}{2||V_k||_\infty}.$$

3-Due to the possible poor approximation of the normal at the zero level set we additionally reinitialized the level set function to the signed distance function, so we solve the problem.

$$\begin{cases} \dfrac{\partial\Phi}{\partial t} + signe(\Phi_0)(|\nabla\Phi| - 1) & = & 0 \\ \Phi(0, x) & = & \Phi_0(x) \end{cases}$$

6. Numerical results

In this section we compare the results of the classical level set method with initial guess (without phase I) to the proposed method that incorporates topological gradient in level set method. We perform all numerical tests on a fixed domain $\Omega = [-1, 1]^2$ with cartesian grid 80 by 80, whichis used to discretize the PDE (5) ,(6), (7) and (8) by the immersed interface method [9] and the Hamilton-Jacobi equation (4).

$$\begin{aligned} \Delta u &= 0 \quad \text{in} \quad \Omega^+ \\ u_{|\Sigma} &= 0 \\ \frac{\partial u}{\partial n}_{|\Gamma} &= g \end{aligned}$$

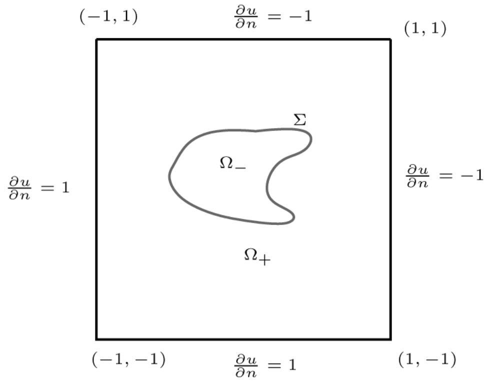

Figure 1. *The partial differential equations and boundary conditions.*

As initial guess we take for all our test exemples a circle with radius $r = 0.7$ and centred at the origin $(0, 0)$.

6.1. Identification of two ellipses

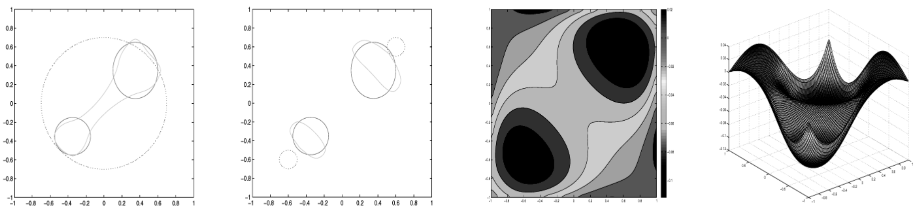

Figure 2. *Left to right: Result using classical level set method of the computed solution after 2000 steps. Result using level set method combined with topological gradient of the computed solution after 100 steps. Isovalues and the plot of topological gradient*

6.2. Identification of for circles

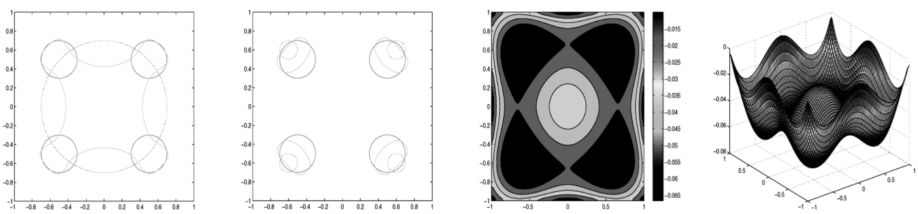

Figure 3. *Left to right: Result using classical level set method of the computed solution after 2000 steps. Result using level set method combined with topological gradient of the computed solution after 100 steps. Isovalues and the plot of topological gradient*

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