Image Segmentation by Riemannian Color Self-Snakes

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RÉSUMÉ. Résumé

ABSTRACT. We present a novel approach for the derivation of PDE modeling curvature-driven flows for color images. We endow the color space with the Helmholtz metric and we derive the differential geometric attributes, such as the covariant derivative and the Christoffel symbols. Then we use these materials to extend scalar-valued mean curvature and snakes methods to the color image setting. Experiments on synthetic image show that the proposed methods are highly robust.

MOTS-CLÉS : Mots clefs

KEYWORDS : Color image, Self-snakes, Segmentation, Riemannian geometry, Helmholtz metric, Mean curvature.

1. Introduction

Throughout this communication, we refer by “segmented image”to an image which has piecewise homogeneous (simplified) regions.

Image segmentation is the first step in image analysis and pattern recognition. It is a critical and essential component of image analysis and one of the most difficult tasks in image processing and determines the quality of the final result of analysis.
The segmentation techniques for monochrome images can be extended to segment color images by using R, G and B or their transformations (linear/non linear) [7, 8, 9, 10].

The generalization of the PDEs methods used for Gray level image to color image is being pursued with mainly three techniques :(1) application of Gray level methods directly to each component of a color space, then the results can be combined in some way to obtain a final restoration result. However, one of the problems is how to employ the color information as a whole for each pixel. (2) the use of Di Zenzo's concept of a structure tensor to create a dependance between color channels ; and (3) differential-geometric methods.

The aim of the present paper is to give a natural generalization of curvature driven methods, like the mean curvature motion (MCM), modified mean curvature flow and self snakes, to color. The key ingredient for these generalizations is the use of the Riemannian geometry of the color space.

We use the fact that when we come to deal with general (warped) spaces, the appropriate initial concept on which to base all geometry is that of the scalar product operation on pairs of tangent vectors [3]. Then, we can perform the measurements on the manifold like lengths and angles in terms of the scalar product. The concept of length determines what we call metric. In the numerical experiments section, we have used the affine metric which is, in the context of color perception, nothing but the Helmholtz metric.

2. Differential Geometry of the Helmholtz Color Space

Hermann von Helmholtz (1821-1894), was the first who had attempted to mathematically formulate the distance between colors by the concept of line element. He define the following line element :

$$ds^2 = \left(\frac{dR}{R}\right)^2 + \left(\frac{dG}{G}\right)^2 + \left(\frac{dB}{B}\right)^2, \tag{1}$$

where R, G and B are the three color channels : Red, Green and blue. In local coordinates, this can be expressed as a positive definite symmetric matrix :

$$(g_{ij})_{i,j=1,2,3} = \begin{bmatrix} \frac{1}{x_1^2} & 0 & 0 \\ 0 & \frac{1}{x_2^2} & 0 \\ 0 & 0 & \frac{1}{x_3^2} \end{bmatrix}, \tag{2}$$

where we use the coordinate notation $x_1 = R, x_2 = B$ and $x_3 = G$.

The color space is defined as a domain Ω in the positive orthant $\mathbb{R}^3_+$ defined by :

$$\mathbb{R}^3_+ = \left\{x \in \mathbb{R}^3 | \quad x_i > 0, \quad i = 1, 2, 3\right\} \tag{3}$$

We remark that the introduction of the Helmholtz metric, which is affine, is equivalent to the nonlinear coordinate transformation $u_i = \log(x_i), i = 1, 2, 3$, with the Jacobian matrix

$$J\mathbf{u}(\mathbf{x}) = \begin{bmatrix} \frac{1}{x_1} & 0 & 0 \\ 0 & \frac{1}{x_2} & 0 \\ 0 & 0 & \frac{1}{x_3} \end{bmatrix}, \quad \mathbf{x} \in \mathbb{R}^3, \tag{4}$$

and that a nonlinear coordinate transformation corresponds to the introduction of a Riemannian metric given by the Jacobian matrix in the form of $G(\mathbf{x}) = J\mathbf{u}(\mathbf{x})^T J\mathbf{u}(\mathbf{x})$ on the definition domain[6]. We note that the color space is geodesic convex with respect to the helmholtz metric.

Having the expression of the metric, we can now give the 3^3 Christoffel symbols using the formula :

$$\Gamma^i_{jk} = \frac{1}{2} g^{ij} \left(\frac{\partial g_{jl}}{\partial x_k} + \frac{\partial g_{kl}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_l} \right), \tag{5}$$

and hence

$$\Gamma^1(\mathbf{x}) = \begin{bmatrix} -\frac{1}{x_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Gamma^2(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{x_2} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Gamma^3(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{x_3} \end{bmatrix}, \mathbf{x} \in \mathbb{R}^3.$$

We can derive now the differential equation of the geodesic arcs which is equal to

$$x''(s) = \frac{1}{x_i(s)} x'_i(s)^2, \quad s \in [s_1, s_2], \quad i = 1, 2, 3, \tag{6}$$

i.e.,

$$\left(\frac{x_i(s)}{x'_i(s)} \right)' = 0, \quad s \in [s_1, s_2], \quad i = 1, 2, 3, \tag{7}$$

where s is the arc-length parameter. It follows that the solution $\gamma(s) = (x_1(s), x_2(s), x_3(s))$ is defined on the whole $\mathbb{R}$ and

$$x_i(s) = e^{a_i s + b_i}, \quad s \in \mathbb{R}, \quad i = 1, 2, 3. \tag{8}$$

If we consider two arbitrary points $\mathbf{x}, \mathbf{y}$ in the Helmholtz color space, then a geodesic joining them is

$$\begin{aligned} \gamma(s) &= (x_1 e^{(\log(y_1) - \log(x_1))s}, x_2 e^{(\log(y_2) - \log(x_2))s}, x_3 e^{(\log(y_3) - \log(x_3))s}) \\ &= (x_1^{1-s} y_1^s, x_2^{1-s} y_2^s, x_3^{1-s} y_3^s), \end{aligned} \qquad s \in [0, 1], \tag{9}$$

because $\gamma(0) = \mathbf{x}$, $\gamma(1) = \mathbf{y}$ and the coordinate functions are positive, so their values are completely included in $\mathbb{R}^3_+$.

3. Immersions and Mean Curvature

Now let $\phi : M \to N$ be an immersion of a manifold M into a Riemannian manifold N with metric g.The mapping ϕ induces a metric $\phi^* g$ on M defined by

$$\phi^* g\left(X_p, Y_p\right) = g\left(\phi_*(X_p), \phi_*(Y_p)\right). \tag{10}$$

This metric is called the *pull-back* metric induced by ϕ, as it maps the metric in the opposite direction of the mapping ϕ. An isometry is a diffeomorphism $\phi : M \to N$ that preserves the Riemannian metric, i.e., if γ and g are the metrics for M and N, respectively, then $\gamma = \phi^* g$.
The first fundamental form associated with the immersion ϕ is $h = \phi^* g$. Its components are $h_{\alpha\beta} = \partial_\alpha \phi^i \partial_\beta \phi^j g_{ij}$ where $\partial_\alpha \phi^i = \frac{\partial \phi^i}{\partial x^\alpha}$. The total covariant derivative $\nabla d\phi$ is called the second fundamental form of ϕ and is denoted by $II^M(\phi)$. The second fundamental form II^M takes values in the normal bundle of M. The mean curvature vector $\boldsymbol{H}$ of an isometric immersion $\phi : M \to N$ is defined as the trace of the second fundamental form $II^M(\phi)$ divided by $m = \dim M$ [4]. In the case of space color $m = 3$.

$$\boldsymbol{H} := \frac{1}{m} \operatorname{tr}_\gamma II^M(\phi). \tag{11}$$

In local coordinates, we have [4]

$$mH^i = \Delta_M \phi^i + \gamma^{\alpha\beta}(x)^N \Gamma^i_{jk}\left(\phi(x)\right) \frac{\partial \phi^j}{\partial x^\alpha} \frac{\partial \phi^k}{\partial x^\beta}. \tag{12}$$

where ${}^N\Gamma^i_{jk}$ are the Christoffel symbols of (N, g) and Δ_M is the Laplace-Beltrami operator on (M, γ) given by

$$\Delta_M = \frac{1}{\sqrt{\det \gamma}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\det \gamma} \gamma^{\alpha\beta} \frac{\partial \phi^i}{\partial x^\beta} \right). \tag{13}$$

4. Geometric Curvature-Driven Flows for Color images

The basic concept in which geometric curvature-driven flows are based is the mean curvature of a submanifold embedded in a higher dimensional manifold. Here we generalize the scalar mean curvature flow to mean curvature flow in the space-feature manifold. For this, we embed the Euclidean image space Ω into the Riemannian manifold $\Omega \otimes \mathbb{R}^3_+$, and use some classical results from differential geometry to derive the *Riemannian Mean Curvature* (RMC). We then use the RMC to generalize mean curvature flow to the Vector-valued data. Given the expression of the mean curvature vector $\boldsymbol{H}$, we can establish some PDEs based Color-image segmentation. Especially, we are interested of the so called level-set methods, which relay on PDEs that modify the shape of level sets in an image.

4.1. Minimal Surface Flow

The motion by the mean curvature vector $\boldsymbol{H}$ yields the Minimal Surface flow :

$$\partial_t \phi^i = H^i. \tag{14}$$

This flow can be considered as a deformation of the Color field toward *minimal immersion*. Indeed, it derives from variational setting that minimize the *volume* of the embedded image manifold in the space-feature manifold.

4.2. Riemannian Mean Curvature Flow

The following flow was proposed for the processing of scalar-valued images

$$\partial_t u = |\nabla u| \operatorname{div} \frac{\nabla u}{|\nabla u|}, \quad u(0,x,y) = u_0(x,y), \tag{15}$$

where $u_0(x,y)$ is the Grey level of the image to be processed, $u(t,x,y)$ is its smoothed version that depends on the scale parameter t.

The "philosophy" of this flow is that the term $|\nabla u| \operatorname{div} \frac{\nabla u}{|\nabla u|}$ represents a degenerate diffusion term which diffuses u in the direction orthogonal to its gradient ∇u and does not diffuse at all in the direction of ∇u.

This formulation has been proposed as a "morphological scale space" [2] and as more numerically tractable method of solving total variation.

The natural generalization of this flow to Color image is

$$\partial_t \phi^i = |\nabla^\gamma \phi|_g H^i, \quad i = 1, \ldots, d, \quad where \quad \nabla^\gamma \phi^i = (\gamma_{ij})^{-1} \nabla \phi^i \tag{16}$$

and

$$|\nabla^\gamma \phi|_g = \gamma^{\alpha\beta} g_{ij} \partial_\alpha \phi^i \partial_\beta \phi^j.$$

We note that several authors have tried to define the norm of vector-image. Most of them use some generalization of the De Zenzo norm. We think that the former equation is the correct generalization, and what we find in the litterature are particular cases done by particular choices of the feature-space metric.

4.3. Modified Riemannian Mean Curvature Flow

To denoise highly degraded images, Alvarez et al. [1] have proposed a modification of the mean curvature flow equation (15) that reads

$$\partial_t \phi = c\left(|K \star \nabla \phi|\right) |\nabla \phi| \operatorname{div} \frac{\nabla \phi}{|\nabla \phi|}, \quad \phi(0,x,y) = \phi_0(x,y), \tag{17}$$

where K is a smoothing kernel (a Gaussian for example), $K \star \nabla \phi$ is therefore a local estimate of $\nabla \phi$ for noise elimination, and $c(s)$ is a nonincreasing real function which tends to zero as $s \to \infty$. We note that for the numerical experiments we have used $c(|\nabla \phi|) = k^2/(k^2 + |\nabla \phi|^2)$.

The generalization of the modified mean curvature flow to color image processing is

$$\partial_t \phi^i = c\left(|K \star \nabla^\gamma \phi|_g\right) |\nabla^\gamma \phi|_g H^i, \quad \phi^i(0,\Omega) = \phi_0^i(\Omega), \tag{18}$$

4.4. Riemannan Self-Snakes

The method of Sapiro, which he names *self-snakes* introduces an edge-stopping function into mean curvature flow

$$\begin{aligned} \partial_t \phi &= |\nabla \phi| \operatorname{div} \left(c\left(K \star |\nabla \phi|\right) \tfrac{\nabla \phi}{|\nabla \phi|} \right) \\ &= c\left(K \star |\nabla \phi|\right) |\nabla \phi| \operatorname{div} \left(\tfrac{\nabla \phi}{|\nabla \phi|} \right) + \nabla c\left(K \star |\nabla \phi|\right) \cdot \nabla \phi \end{aligned} \tag{19}$$

Comparing equation (19) to (17), we observe that the term $\nabla c\left(K \star |\nabla \phi|\right) \cdot \nabla \phi$ is missing in the old model. This is due to the fact that the Sapiro model takes into account the image structure. Indeed, equation (19) can be re-written as

$$\partial_t \phi = \mathcal{F}_{\text{diffusion}} + \mathcal{F}_{\text{shock}}, \tag{20}$$

where

$$\mathcal{F}_{\text{diffusion}} = c\left(K \star |\nabla\phi|\right)|\nabla\phi|\,\text{div}\left(\frac{\nabla\phi}{|\nabla\phi|}\right),$$

$$\mathcal{F}_{\text{shock}} = \nabla c\left(K \star |\nabla\phi|\right)\cdot\nabla\phi.$$

The term $\mathcal{F}_{\text{diffusion}}$ is as in the anisotropic flow proposed in [1]. The second term in (20), i.e., $\nabla c \cdot \nabla\phi$, increases the attraction of the deforming contour toward the boundary of "objects" acting as the shock-filter for deblurring. Therefore, the flow $\nabla c \cdot \nabla\phi$ is a shock filter acting like the backward diffusion in the Perona-Malik equation, which is responsible for the edge-enhancing properties of self snakes. See [7] for detailed discussion on this topic.

We are now interested in generalizing Self-Snakes method for the case of Color image. We will start the generalization from equation (20) in the following manner

$$\partial_t\phi = \mathcal{F}_{diffusion} + \mathcal{F}_{shock}, \tag{21}$$

where

$$\begin{array}{lcl} \mathcal{F}_{diffusion} & = & c\left(K \star |\nabla^{\gamma}\phi|_g\right)|\nabla^{\gamma}\phi|_g H^i \\ \mathcal{F}_{shock} & = & \nabla c\left(K \star |\nabla^{\gamma}\phi|_g\right)\cdot\nabla^{\gamma}\phi^i. \end{array} \tag{22}$$

5. Numerical Experiments

We give numerical experiments only for the self snakes method since it is the most general case.

Figure 1. *Original image (left) Highly degraded image (center) Segmented image (right)*

In this paper we generalized several curvature-driven flows of Gray level image to Color image. The use of the differential-geometric tools and concepts yields a natural extension of these well-known scalar-valued data processing methods to vector-valued data processing.

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