Analysis of piecewise polynomial DDFV solutions for subsurface flow problems

Analysis of DDFV solutions to flow problems

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RÉSUMÉ. L'objectif de cette communication est de présenter les concepts de solutions approchées localement P_1 (i.e. linéaire), Q_1 (i.e. bilinéaire) et Q_2 (i.e. biquadratique) dans le cadre d'une analyse volumes finis des problèmes de diffusion en milieux poreux hétérogènes anisotropes. L'introduction de ces concepts est inspirée par l'interpolation lagrangienne utilisée en théorie d'éléments finis. On déduit certaines estimations d'erreur pour ces solutions approchées à partir d'estimations d'erreur sur la solution du problème discrèt.

ABSTRACT. The aim of this communication is to present, in the framework of finite volume analysis of diffusion problems, the concepts of continuous piecewise linear (i.e. P_1), continuous piecewise bilinear (i.e. Q_1) and continuous piecewise biquadratic (i.e. Q_2) approximate solutions to flow problems in anisotropic nonhomogeneous porous media. We derive the convergence in L^2-norm of these approximate solutions from some error estimates obtained for the solution of the discrete problem.

MOTS-CLÉS : Ecoulements souterrains, anisotropie, hétérogénéité, volumes finis, solutions approchées localement polynomiales, estimations d'erreur.

KEYWORDS : Subsurface flows, anisotropy, heterogeneities, finite volumes, cellwise polynomial solutions, error estimates.

1. Introduction and the model problem

The finite volume analysis on general grids of flow problems in anisotropic nonhomogeneous porous media has been widely developed in the mathematical literature these last years (see for instance [1,2,3,6,7,8,10,11] and the references therein). An example of such problems reads as follows : Find a function φ defined in $\overline{\Omega}$, which satisfies the following partial differential equation associated with homogeneous Dirichlet boundary conditions :

$$-div(D\, grad\, \varphi) = f \quad \text{in} \quad \Omega \tag{1.1}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma \tag{1.2}$$

where f is a given function (commonly called source/sink term), Ω is a given open polygonal domain and Γ denotes its boundary. $D = D(x)$, with $x = (x_1, x_2)^t \in \Omega$, is a full symmetric matrix describing the spatial variation of the diffusion coefficient which satisfies to the uniform ellipticity i.e.

$$\exists \gamma \in \mathbb{R}_+^* \quad \text{such that} \ \forall \xi \in \mathbb{R}^2, \xi \neq 0 \quad \gamma\ |\xi|^2 \leq \xi^t D(x) \xi \ \ a.e. \ in \ \Omega \tag{1.3}$$

where $|\,.\,|$ denotes the euclidian norm in $\mathbb{R}^2$. The components $D_{ij}(.)$ of the diffusion matrix $D(.)$ are $L^\infty(\Omega)$-functions. For $f \in L^2(\Omega)$ one easily proves that this problem gets a unique variational solution in the well known Sobolev $H_0^1(\Omega)$ (see for instance [4] for more details). Many methods have been proposed for addressing the flow problem (1.1)-(1.2) in the framework of finite volume methods. Among those methods the most investigated today are Multi-Point Flux Approximation - MPFA for short - (see the pioneer work from [1]), Discrete Duality Finite Volumes - DDFV for short - (see the pioneer work in [6]) and MPFA of DDFV type (see the pioneer works of [7,10]).

Our talk concerns a DDFV analysis of single-phase subsurface flow problems governed by full permeability tensors. Applying this method to (1.1)-(1.2) leads to local flux computations involving cell center pressures, cell vertex pressures and edge midpoint pressures. Two main options are possible to define the corresponding discrete system : one option consists in expressing the edge midpoint pressures in terms of linear function of cell center and cell vertex pressures, in view to focus on a linear discrete system involving only cell center and cell vertex pressures as discrete unknowns ; the other option is based upon treating all the pressures as discrete unknowns of the same importance. Following the spirit of the finite element theory in 2D cartesian meshes, the first option permits to introduce the concept of P_1-finite volume solutions, while the second option allows to develop the concepts of Q_1- and Q_2- finite volume solutions.

Our purpose in this communication goes in two directions : (i) Presenting an overview of the ideas behind the above concepts completely new in the finite volume framework ; (ii) Dealing with stability and error estimates for the concept of piecewise polynomial DDFV solutions. For this end our talk is organized as follows. In the next section, we give a DDFV formulation of the continuous problem (1.1)-(1.2), followed by the existence and uniqueness result for the discrete solution. We introduce in the third section the concepts of P_1-, Q_1- and Q_2-finite volume solutions following the ideas developed in the earlier work from [9]. The fourth section is devoted to the stability and error estimates for discrete solutions. In the fifth section, we focus on the L^2 convergence of P_1-, Q_1- and Q_2- approximate solutions of (1.1)-(1.2). The last section concerns concluding comments.

2. DDFV discretization of (1.1)-(1.2). Existence and uniqueness of discrete solutions

Let us suppose that the diffusion matrix $D(.)$ involved in the balance equation (1.1) is a piecewise constant full tensor. As one can see in [5] this assumption is current in Reservoir Simulation practice. For sake of clarity of the talk, set : $\Omega =]0,1[\times]0,1[$ and define over $\overline{\Omega}$ a square primary grid denoted by $\mathcal{P}$ involving the discontinuities of $D(.)$. Define the size of $\mathcal{P}$ by $h = \frac{1}{N}$, where N is a given strictly positive integer. Note that some co-authors of this talk have analyzed numerically and theoretically in [8,11,12] the convergence of a DDFV solution to (1.1)-(1.2), with Ω being a general polygon over which is defined an unstructured primary grid. The interest of this work lies in the introduction and analysis of the concept of piecewise polynomial DDFV solutions.

2.1. Formulation of the DDFV discrete problems

We give in this subsection the matrix form of the DDFV formulation for (1.1)-(1.2) in two cases : (i) Without edge midpoint approximate pressures, (ii) With edge midpoint approximate pressures.

We denote by K_{ij} the primary grid-block defined by : $K_{ij} = \left[x_1^{i-\frac{1}{2}}, x_1^{i+\frac{1}{2}}\right] \times \left[x_2^{j-\frac{1}{2}}, x_2^{j+\frac{1}{2}}\right]$, where $x_1^{i+\frac{1}{2}} = x_1^{i-\frac{1}{2}} + h,\ \ x_2^{j+\frac{1}{2}} = x_2^{j-\frac{1}{2}} + h,\ for\ i,j = 1,\dots,N$ with $x_1^{\frac{1}{2}} = x_2^{\frac{1}{2}} = 0.$

In what follows we assume that the solution φ of (1.1)-(1.2) is sufficiently regular for our purpose.

• First case : a discrete problem with cell center and cell vertex pressures as discrete unknowns. We are looking for a linear system involving $U_{cc} = \{u_{i,j}\}_{1\le i,j\le N}$ and $U_{vc} = \left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ as discrete unknowns expected to be reasonable approximations of $\{\varphi_{i,j}\}_{1\le i,j\le N}$ (cell center pressures) and $\left\{\varphi_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\le i,j\le N-1}$ (cell vertex pressures) respectively, where $\varphi_{i,j} = \varphi\left(x_1^i, x_2^j\right)$ and $\varphi_{i+\frac{1}{2},j+\frac{1}{2}} = \varphi\left(x_1^{i+\frac{1}{2}}, x_2^{j+\frac{1}{2}}\right)$, with :

$$x_1^i = \frac{x_1^{i-\frac{1}{2}} + x_1^{i+\frac{1}{2}}}{2}, \qquad x_2^j = \frac{x_2^{j-\frac{1}{2}} + x_2^{j+\frac{1}{2}}}{2} \qquad 1 \le i\,,j\ \le N \tag{2.1}$$

We also adopt the following conventions :

$$x_1^0 = x_1^{\frac{1}{2}}, \quad x_1^{N+1} = x_1^{N+\frac{1}{2}}, \quad x_2^0 = x_2^{\frac{1}{2}}, \quad x_2^{N+1} = x_2^{N+\frac{1}{2}} \tag{2.2}$$

We give now a summary description of the procedure leading to the linear discrete system without edge midpoint pressures (see [9] for details). Integrating the balance equation (1.1) in the primary grid-block K_{ij} and applying the Ostrogradski's theorem leads to compute the flux on the boundary of K_{ij}. Performing this computation with an adequate quadrature formula over each half-edge of K_{ij} yields an expression involving the edge midpoint pressures. Thanks to the flux continuity over the primary grid-block interfaces, edge midpoint pressures may be expressed in terms of cell and vertex pressures . Similarly, one integrates the balance equation (1.1) in dual grid-blocks defined by : $K_{i+\frac{1}{2},j+\frac{1}{2}} = \left[x_1^i, x_1^{i+1}\right] \times \left[x_2^j, x_2^{j+1}\right]$ and performs (in the same way than the primary grid-block case) the flux computation over the boundary of $K_{i+\frac{1}{2},j+\frac{1}{2}}$. Therefore one deduces a discrete problem which consists in finding $U_{cc} = \{u_{i,j}\}_{1\leq i,j\leq N}$ and $U_{vc} = \left\{u_{i+\frac{1}{2},j+\frac{1}{2}}\right\}_{1\leq i,j\leq N-1}$ such that :

$$\begin{aligned}
&\tfrac{2D_{22}^{ij}D_{22}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[u_{i,j}-u_{i,j+1}\right]+\tfrac{D_{22}^{ij}D_{21}^{ij+1}+D_{22}^{ij+1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[u_{i-\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{2D_{22}^{ij}D_{22}^{ij-1}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[u_{i,j}-u_{i,j-1}\right]+\tfrac{D_{22}^{ij}D_{21}^{ij-1}+D_{22}^{ij-1}D_{21}^{ij}}{D_{22}^{ij}+D_{22}^{ij-1}}\left[u_{i+\frac{1}{2},j-\frac{1}{2}}-u_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\tfrac{2D_{11}^{ij}D_{11}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[u_{i,j}-u_{i+1,j}\right]+\tfrac{D_{11}^{ij}D_{12}^{i+1j}+D_{11}^{i+1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[u_{i+\frac{1}{2},j-\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{2D_{11}^{ij}D_{11}^{i-1j}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[u_{i,j}-u_{i-1,j}\right]+\tfrac{D_{11}^{ij}D_{12}^{i-1j}+D_{11}^{i-1j}D_{12}^{ij}}{D_{11}^{ij}+D_{11}^{i-1j}}\left[u_{i-\frac{1}{2},j+\frac{1}{2}}-u_{i-\frac{1}{2},j-\frac{1}{2}}\right]\\
&=\int_{K_{ij}}f(x)dx \qquad \forall\, 1\leq\ i,\ j\ \leq\ N
\end{aligned} \tag{2.3}$$

$$\begin{aligned}
&\tfrac{D_{11}^{ij+1}D_{21}^{i+1j+1}+D_{11}^{i+1j+1}D_{21}^{ij+1}}{D_{11}^{ij+1}+D_{11}^{i+1j+1}}\left[u_{i,j+1}-u_{i+1,j+1}\right]+\\
&\left(\tfrac{\left(D_{12}^{i+1j+1}-D_{12}^{ij+1}\right)\left(D_{21}^{ij+1}-D_{21}^{i+1j+1}\right)}{2\left(D_{11}^{ij+1}+D_{11}^{i+1j+1}\right)}+\tfrac{D_{22}^{ij+1}+D_{22}^{i+1j+1}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j+\frac{3}{2}}\right]\\
&+\tfrac{D_{11}^{i+1j}D_{21}^{ij}+D_{11}^{ij}D_{21}^{i+1j}}{D_{11}^{ij}+D_{11}^{i+1j}}\left[u_{i+1,j}-u_{i,j}\right]+\\
&\left(\tfrac{\left(D_{12}^{i+1j}-D_{12}^{ij}\right)\left(D_{21}^{ij}-D_{21}^{i+1j}\right)}{2\left(D_{11}^{ij}+D_{11}^{i+1j}\right)}+\tfrac{D_{22}^{ij}+D_{22}^{i+1j}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{1}{2},j-\frac{1}{2}}\right]\\
&+\tfrac{D_{22}^{i+1j}D_{12}^{i+1j+1}+D_{22}^{i+1j+1}D_{12}^{i+1j}}{D_{22}^{i+1j}+D_{22}^{i+1j+1}}\left[u_{i+1,j}-u_{i+1,j+1}\right]+\\
&\left(\tfrac{\left(D_{21}^{i+1j+1}-D_{21}^{i+1j}\right)\left(D_{12}^{i+1j}-D_{12}^{i+1j+1}\right)}{2\left(D_{22}^{i+1j}+D_{22}^{i+1j+1}\right)}+\tfrac{D_{11}^{i+1j}+D_{11}^{i+1j+1}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i+\frac{3}{2},j+\frac{1}{2}}\right]\\
&+\tfrac{D_{22}^{ij+1}D_{12}^{ij}+D_{22}^{ij}D_{12}^{ij+1}}{D_{22}^{ij}+D_{22}^{ij+1}}\left[u_{i,j+1}-u_{i,j}\right]+\\
&\left(\tfrac{\left(D_{21}^{ij+1}-D_{21}^{ij}\right)\left(D_{12}^{ij}-D_{12}^{ij+1}\right)}{2\left(D_{22}^{ij}+D_{22}^{ij+1}\right)}+\tfrac{D_{11}^{ij}+D_{11}^{ij+1}}{2}\right)\left[u_{i+\frac{1}{2},j+\frac{1}{2}}-u_{i-\frac{1}{2},j+\frac{1}{2}}\right]\\
&=\int_{K_{i+\frac{1}{2},j+\frac{1}{2}}}f(x)dx \qquad \forall\, 1\leq\ i,\ j\ \leq\ N-1
\end{aligned} \tag{2.4}$$

where we have set

$$u_{i+\frac{1}{2},\frac{1}{2}} = u_{i+\frac{1}{2},N+\frac{1}{2}} = u_{\frac{1}{2},j+\frac{1}{2}} = u_{N+\frac{1}{2},j+\frac{1}{2}} = 0 \quad \forall\ \ 0\leq\ i,\ j\leq\ N \tag{2.5}$$

and

$$u_{i,0} = u_{0,j} = u_{i,N+1} = u_{N+1,j} = 0 \qquad \forall\ \ 1\leq\ i,\ j\leq\ N \tag{2.6}$$

The previous discrete problem may be put in the following matrix form :

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \end{bmatrix} = \begin{bmatrix} F_{cc} \\ F_{vc} \end{bmatrix} \tag{2.7}$$

• Second case : a discrete problem involving cell center pressures, cell vertex pressures and edge midpoint pressures as discrete unknowns. This discrete problem is a system made up of flux continuity equations over primary grid-block interfaces and discrete balance equations associated respectively with primary grid-blocks and dual grid-blocks (we have introduced above). The matrix form for this discrete problem reads :

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} U_{cc} \\ U_{vc} \\ U_{ep} \end{bmatrix} = \begin{bmatrix} F_{cc} \\ F_{vc} \\ 0 \end{bmatrix} \tag{2.8}$$

Note that the discrete problem (2.7) is actually derived from the discrete problem (2.8) after elimination of the edge midpoint pressures.

2.2. Existence and uniqueness for solutions of discrete problems

Following [9] one can prove that :

Proposition 2.1 *The discrete problems (2.7) and (2.8) possess unique discrete solutions·* ◇

Proposition 2.2 *The matrices associated respectively to the discrete problems (2.7) and (2.8) are positive definite.*
Moreover, the matrix $\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ *is symmetric·* ◇

3. Introduction of three families of approximate solutions in terms of continuous functions

Solving the discrete problem (2.8) leads to determining cell center pressures, cell vertex pressures and edge midpoint pressures (with respect to the primary grid). But solving the discrete problem (2.7) yields only cell center pressures and cell vertex pressures.

3.1. The class of continuous piecewise linear approximate solutions

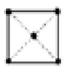

Figure 1. *A primary grid block divided into four triangular elements T. The symbol • represents degrees of freedom (i.e. nodal values given as solution of the discrete problem (2.7)) for the linear approximate solution over triangular elements.*

We start by dividing each primary grid-block into four triangular elements, with generic name T, constructed by joining the center of that primary grid-block to its four vertices (see Figure 1 above). By doing so, one generates over Ω a new grid denoted by $\mathcal{T}$. Let $U_{\mathcal{T}}$ be the piecewise linear approximate solution associated with the grid $\mathcal{T}$. It is the so-called P_1-finite volume solution.

3.2. The Class of piecewise bilinear approximate solutions

We start with dividing each square primary grid-block into four uniform square elements, with generic name S (see Figure 2 below). By doing so, one generates over Ω a new grid denoted by $\mathcal{S}$. Let us denote by $U_{\mathcal{S}}$ the piecewise bilinear approximate solution associated with the grid $\mathcal{S}$. It is the so-called Q_1-finite volume solution.

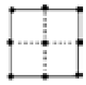

Figure 2. *A primary grid block divided into four square elements S for a piecewise bilinear approximation of the solution. The symbol • represents degrees of freedom (i.e. nodal values given as solution of the discrete problem (2.7)) for the bilinear approximate solution over square elements.*

3.3. The Class of piecewise biquadratic approximate solutions

Let us denote by $U_{\mathcal{P}}$ the piecewise biquadratic approximate solution associated with the primary grid $\mathcal{P}$ (see Figure 3 below). It is the so-called Q_2-finite volume solution.

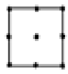

Figure 3. *The symbol • represents degrees of freedom (i.e. nodal values given as solution of the discrete problem (2.8)) for the biquadratic approximate solution over primary grid-blocks.*

We have the following result which is easy to prove using arguments from finite element theory.

Proposition 3.1 *The approximate solutions $U_{\mathcal{T}}$, $U_{\mathcal{S}}$ and $U_{\mathcal{P}}$ are continuous functions in $\overline{\Omega}$ (closure of Ω). Moreover these approximate solutions belong to the same functional space as the exact solution, namely the space $H_0^1(\Omega)$ which is a well-known Sobolev space.*$\diamond$

4. Stability and error estimates for discrete solutions

The introduction of two discrete energy norms is needed before formulating stability and error estimates results. For that purpose let us consider two other grids :
(i) The grid $\mathcal{L}$ which is made up of uniform rhombi centered on cell centers and cell vertices of the primary square mesh (except boundary vertices). To $\mathcal{L}$ is attached the space $\mathbf{E}(\mathcal{L})$ made up of constant functions over the rhombi and zero elsewhere in Ω. Let us endow $\mathbf{E}(\mathcal{L})$ with the following discrete energy norm. For all $v \in \mathbf{E}(\mathcal{L})$ we set :

$$\|v\|_{1,h}^2 = \sum_{s \in S} |\Delta_s v|^2, \quad \text{with } |\Delta_s v|^2 = \sum_{\substack{L,K \in \mathcal{L} \text{ such that} \\ \Gamma_K \cap \Gamma_L = \{s\}}} |v_L - v_K|^2 \tag{4.1}$$

where S is the set of rhombi vertices, $|\Delta_s v|^2 = |v_L|^2$, if $s \in \Gamma \cap L$, with $L \in \mathcal{L}$.
(ii) The grid $\mathcal{M}$ which is made up of uniform square elements of size $\frac{h}{2}$, centered on the nodes (recall that a node is a cell center, a cell vertex or an edge midpoint from the primary mesh) and completely imbedded in $\overline{\Omega}$. With this grid is associated **E**($\mathcal{M}$) the space of functions v constant in every $M \in \mathcal{M}$ and zero elsewhere in Ω. We equip this space with the discrete energy norm defined by

$$\|v\|_{2,h}^2 = \sum_{p\,,\,q \in \mathcal{N},\ d(p,q)=\frac{h}{2}} |v_p \,-\, v_q|^2 \tag{4.2}$$

where $\mathcal{N}$ is the set of nodes, and where v_p and v_q are constant values of v in the square elements from $\mathcal{M}$ centered respectively on the nodes p and q.
Let us set : $u_h = (U_{cc}, U_{vc})$, $\tilde{u}_h = (U_{cc}, U_{vc}, U_{ep})$, $\varphi_h = (\varphi_{cc}, \varphi_{vc})$, $\tilde{\varphi}_h = (\varphi_{cc}, \varphi_{vc}, \varphi_{ep})$, $\varepsilon_h = \varphi_h - u_h$, $\tilde{\varepsilon}_h = \tilde{\varphi}_h - \tilde{u}_h$. Then, we have the following estimates (see [9]), where C represents diverse constants strictly positive and mesh independent.

Proposition 4.1 *(Stability results for discrete solutions)*

$$\|u_h\|_{1,h} \le C\,\|f\|_{L^2(\Omega)} \qquad \textit{and} \qquad \|\tilde{u}_h\|_{1,h} \le C\,\|f\|_{L^2(\Omega)} \cdot \diamond$$

Proposition 4.2 *(Error estimates for discrete solutions)*

$$\|\varepsilon_h\|_{1,h} \le C\,h, \qquad \|\varepsilon_h\|_{L^\infty(\Omega)} \le C\,h^{\frac{1}{2}}, \qquad \|\varepsilon_h\|_{L^2(\Omega)} \le C\,h \,\cdot$$

$$\|\tilde{\varepsilon}_h\|_{1,h} \le C\,h, \qquad \|\tilde{\varepsilon}_h\|_{L^\infty(\Omega)} \le C\,h^{\frac{1}{2}}, \qquad \|\tilde{\varepsilon}_h\|_{L^2(\Omega)} \le C\,h \cdot \diamond$$

Note that in the earlier work [11], the following results have been proven : If Ω is anisotropic but homogeneous and the exact solution φ of (1.1)-(1.2) is in $C^3(\overline{\Omega})$ then

$$\|\varepsilon_h\|_{1,h} \le C\,h^2, \;\; \|\varepsilon_h\|_{L^\infty(\Omega)} \le C\,h^{\frac{3}{2}} \;\textit{and}\; \|\varepsilon_h\|_{L^2(\Omega)} \le C\,h^2 \cdot \diamond \tag{4.3}$$

5. L^2-convergence for $U_{\mathcal{T}}$, $U_{\mathcal{S}}$ and $U_{\mathcal{P}}$

The L^2-Error estimates relevant to the P_1-, Q_1- and Q_2-finite volume solutions to (1.1)-(1.2) can be derived from the previous estimates as proven in [9]. More precisely, we have :

Proposition 5.1 *Assume that the permeability tensor D in the flow problem (1.1)-(1.2) is a full matrix which is symmetric and positive definite, with piecewise constant coefficients. Assume also that the discontinuities of D coincide with primary mesh interfaces and divide Ω into subdomains $(\Omega_s)_{s\in S}$ such that the unique variational solution φ of (1.1)-(1.2) satisfies to $\varphi_{|\overline{\Omega_s}} \in C^2\left(\overline{\Omega_s}\right)$, for all $s \in S$. Then, φ, $U_{\mathcal{T}}$, $U_{\mathcal{S}}$ and $U_{\mathcal{P}}$ satisfy to the following error estimates :*

$$\|\varphi - U_{\mathcal{T}}\|_{L^2(\Omega)} \le C\,h, \;\; \left\|\varphi - U_{\mathcal{S}}^h\right\|_{L^2(\Omega)} \le C\,h, \;\; \left\|\varphi - U_{\mathcal{P}}^h\right\|_{L^2(\Omega)} \le C\,h. \diamond \tag{5.1}$$

6. Concluding comments

DDFV analysis of (1.1)-(1.2) on general meshes defined over a polygonal domain with piecewise constant permeability tensor have been carried out by our team. More precisely, Njifenjou, Mbehou and Moukouop-Nguena have proven in [12] stability and convergence of the DDFV solution for Dirichlet boundary conditions. In forthcoming papers, Kinfack and Njifenjou have shown stability and convergence of the DDFV solution for Neumann boundary conditions, Mbehou and Njifenjou have obtained similar results for periodic boundary conditions. Numerical tests on FVCA5 Benchmark problems confirmed the theoretical results (see [8][12]).

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7. Bibliographie

[1] I. AAVATSMARK, T. BARKVE, O. BOE, T. MANNSETH, « Discretization on non-orthogonal curvilinear grids for multi-phase flow », *Proceedings of the Fourth European Conference on the Mathematics of Oil Recovery*, Roros, Norway, 7-10 June 1994.

[2] B. ANDREIANOV, F. BOYER, F. HUBERT, « Discrete duality finite volume schemes for Leray-Lions-type elliptic problems on general 2D meshes », *Numer. Methods Partial Differ. Equations*, vol. 23, n° 1, 145-195, 1994.

[3] R. EYMARD, T. GALLOUET, R. HERBIN, « Finite volume methods », *Handbook of Numerical Analysis, VII, Amsterdam (North-Holland), Editors : P.G. Ciarlet, J-L. Lions, pp 713-1020, 2000.*

[4] H. BRÉZIS, « Analyse fonctionnelle , Théorie et applications », *Edited by Masson*, 1983.

[5] L. J. DURLOFSKY, « Upscaling and Gridding of Geologically Complex Systems », *Department of Petroleum Engineering of Stanford University, Chevron Texaco E & P Technology Company.*

[6] K. DOMELEVO, P. OMNES, « A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids », *ESAIM, Math. Model. Numer. Anal.*, vol. 39, n° 6, pp 1203-1249, 1994.

[7] F. HERMELINE, « Approximation of diffusion operators with discontinuous tensor coefficients on distorted meshes », *Comput. Methods Appl. Mech. Eng.*, vol. 192, n° 16-18, pp 1939-1959, 2003.

[8] R. HERBIN, F. HUBERT, « Benchmark on Discretization Schemes for Anisotropic Diffusion Problems on General Grids », *Finite Volume for Complex Applications V, Aussois (France)*, Eds : R. Eymard and J.M. Herard, Wiley, pp 659-692, June 2008.

[9] A. NJIFENJOU, A. J. KINFACK, « Convergence analysis of an MPFA method for flow problems in anisotropic heterogeneous porous media », *International Journal on Finite Volumes*, vol. 5, n° 1, 2008.

[10] I. MOUKOUOP, A. NJIFENJOU, « Traitement numérique des anisotropies de perméabilité dans les écoulements en milieux poreux », *Proceedings of the International Conference on Computer Systems for Environment Management, Douala, October 22-25, 2001*,

[11] A. NJIFENJOU, I.M. NGUENA, « A finite volume approximation for second order elliptic problems with a full matrix on quadrilateral grids : derivation of the scheme and a theoretical analysis », *International Journal on Finite Volumes*, vol. 3, n° 2, 2006 (electronic publication).

[12] A. NJIFENJOU, M. MBEHOU, I. MOUKOUOP NGUENA, « Analysis on general meshes of a DDFV type method for subsurface flow problems », *submitted, revised version.*