Analysis of an Age-structured SIL model

with demographics process and vertical transmission

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ABSTRACT. We consider a mathematical SIL model for the spread of a directly transmitted infectious disease in an age-structured population; taking into account the demographic process and the vertical transmission of the disease. First we establish the mathematical well-posedness of the time evolution problem by using the semigroup approach [4]. Next we prove that the basic reproduction ratio R_0 is given as the spectral radius of a positive operator, and an endemic state exist if and only if the basic reproduction ratio R_0 is greater than unity, while the disease-free equilibrium is locally asymptotically stable if $R_0 < 1$. We also show that the endemic steady states are forwardly bifurcated from the disease-free steady state when R_0 cross the unity. Finally we examine the conditions for the local stability of the endemic steady states.

RÉSUMÉ. Nous considérons ici un modèle mathématique SIL de transmission directe de la maladie dans une population hôte structurée en âge; prenant en compte les processus démographiques et la transmission verticale de la maladie. Premièrement, nous étudions le caractère bien posé du problème par la théorie des semi-groupes [4]. Ensuite, nous montrons que le taux de reproduction de base R_0 est le rayon spectral d'un opérateur positif; et un équilibre endémique existe si et seulement si R_0 est supérieur à l'unité, tandis que l'équilibre sans maladie est localement asymptotiquement stable si $R_0 < 1$. Nous établissons aussi l'existence d'une bifurcation de l'équilibre sans maladie quand R_0 passe par l'unité. Enfin, nous donnons des conditions nécessaires pour la stabilité locale de l'équilibre endémique.

KEYWORDS : Age-structured model, Semigroup, Basic reproduction ratio, Stability.

MOTS-CLÉS : Modèle structuré en âge, Semigroup, Taux de reproduction de base, Stabilité.

1. Introduction

In this paper we consider a mathematical S-I-L (Susceptibles-Infected-Lost of sight) model with demographics process, for the spread of a directly transmitted infectious disease in an age-structured population. The lost of sight class was previously consider in some papers as [1]. We assume that the infective agent can be transmitted not only horizontally but also vertically from adult individuals to their newborns. There are important infective agents such as HBV (hepatitis B virus), HIV (human immunodeficiency virus) and HTLV (human T-cell leukemia virus) that can be vertically transmitted. Compared with the pure horizontal transmission case, so far we have not yet so many results for vertically diseases in structured populations. In Africa, the vertical transmission of the disease like HIV is in progression nowadays.

Firstly, the epidemic system is formulated. Then, we will describe the semigroup approach to the time evolution problem of the abstract epidemic system. Next we consider the disease invasion process to calculate the basic reproduction ratio R_0 [3], then, we prove that the disease-free steady state is locally asymptotically stable if $R_0 < 1$. Subsequently, we show that at least one endemic steady state exists if the basic reproduction ratio R_0 is greater than unity. By introducing a bifurcation parameter, we show that the endemic steady state is forwardly bifurcated from the disease-free steady state when the basic reproduction ratio crosses unity. Finally, we consider the conditions for the local stability of the endemic steady states.

2. The model

In this section, we formulate a model for the spread of the disease in a host population. We consider a host population divided into three subpopulation; the susceptible class, the infective class (those who take a chemoprophylaxis) and the lost of sight class denoted respectively by $S(t,a)$, $I(t,a)$ and $L(t,a)$. Let $\beta(a,\sigma)$ be the transmission rate between the susceptible individuals aged a and the infective or lost of sight individuals aged σ. All recruitment is into the susceptible class and occur at a specific rate $\Lambda(a)$. The rate of non-disease related death is $\mu(a)$. Infected and lost of sight have additional death rates due to the disease $d_1(a)$ and $d_2(a)$ respectively. The transmission of the disease occurs following adequate contacts between a susceptible and infectious or lost of sight. $r(a)$ denoted the proportion of individuals receiving an effective therapy in a care center and $\phi(a)r(a)$ the fraction of them who after begun their treatment will not return in the hospital for the examination. After some time, some of them can return in the hospital at specific rate $\gamma(a)$. The demographics and epidemiology process of the age-structured SIL model is given by Figure 1. The basic system (age-structured SIL epidemic model) with vertical transmission can be formulated as follows by equation (1).

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)S(t,a) &= \Lambda(a)-(\lambda(t,a)+\mu(a))S(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)I(t,a) &= \lambda(t,a)S(t,a)-(\mu(a)+d_1(a) \\ & \quad +r(a)\phi(a))I(t,a)+\gamma(a)L(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right)L(t,a) &= r(a)\phi(a)I(t,a)-(\mu(a)+d_2(a)+\gamma(a))L(t,a), \end{cases} \tag{1}$$

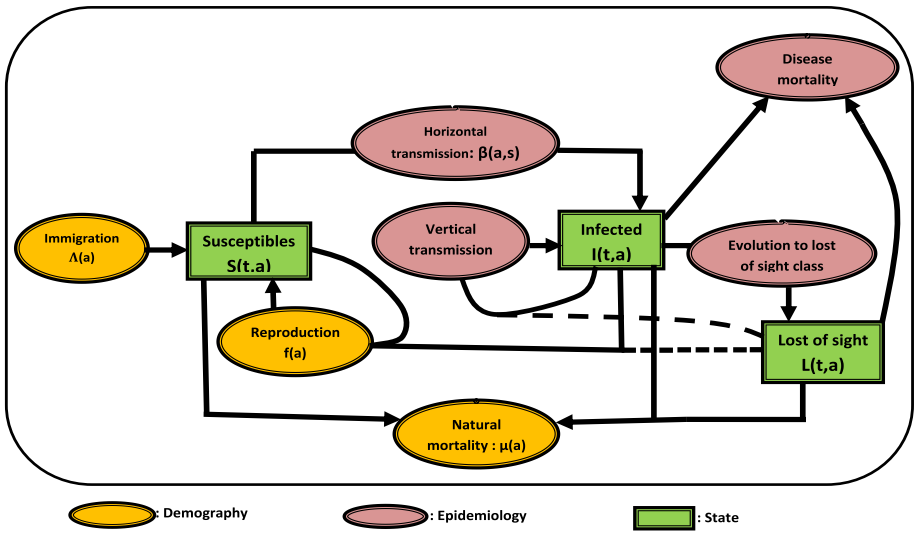


Figure 1: Epidemiology and demographics process of the model.

with initial boundary conditions

$$\left\{\begin{array}{rcl} S(t,0) & = & \int_0^{a^+} f(a)[s(t,a)+(1-p)(I(t,a)+L(t,a))]da, \\ I(t,0) & = & p\int_0^{a^+} f(a)(I(t,a)+L(t,a))da, \\ L(t,0) & = & 0, \\ S(0,a) & = & \varphi_S(a);\ \ a\in(0,a^+), \\ I(0,a) & = & \varphi_I(a);\ \ a\in(0,a^+), \\ L(0,a) & = & \varphi_L(a);\ \ a\in(0,a^+), \end{array}\right. \tag{2}$$

where $f(a)$ is the age-specific fertility rate, p is the proportion of newborns produced from infected individuals who are vertically infected and $a^+ < \infty$ is the upper bound of age. The force of infection $\lambda(t,a)$ is given by

$$\lambda(t,a) = \int_0^{a^+} \beta(a,\sigma)(I(t,\sigma)+L(t,\sigma))d\sigma.$$

Let us write the system in a abstract form. Let $X = L^1(0,a^+)^3$ with the norm $||\varphi||_X = \sum_{i=1}^{3} ||\varphi_i||_{L^1}$, where $\varphi = (\varphi_1,\varphi_2,\varphi_3) \in X$. It is well known that $(X, ||.||_X)$ is a Banach space. Consider the operator $A : D(A) \subset X \to X$ defined by $A\varphi = (-\frac{d\varphi_1}{da}, -\frac{d\varphi_2}{da}, -\frac{d\varphi_3}{da})^T$, where $D(A)$ is the domain of the operator A given by

$$D(A) = \left\{ \varphi=(\varphi_1,\varphi_2,\varphi_3)\in X\ ;\varphi_i\in W^{1,1}(0,a^+)\ and\ \begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\end{pmatrix} = \begin{pmatrix}\int_0^{a^+} f(a)[\varphi_1(a)+(1-p)(\varphi_2(a)+\varphi_3(a))]da\\ p\int_0^{a^+} f(a)(\varphi_2(a)+\varphi_3(a))da\\ 0\end{pmatrix}\right\}$$

and the function $F : \overline{D(A)} \to X$ defined by

$$F\begin{pmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{pmatrix} = \begin{pmatrix}\Lambda - (\lambda[.,\varphi]+\mu)\varphi_1\\ \lambda[.,\varphi]\varphi_1 - (\mu+d_1+r\phi)\varphi_2 + \gamma\varphi_3\\ r\phi\varphi_2 - (\mu+d_2+\gamma)\varphi_3\end{pmatrix},$$

where $\lambda[.,\varphi] \in L^1(0,a^+)$ is such that $\lambda[a,\varphi] = \int_0^{a^+}\beta(a,\sigma)[\varphi_2(\sigma)+\varphi_3(\sigma)]d\sigma$ and $W^{1,1}(0,a^+)$ is a usual Sobolev space. Then the abstract formulation of the system (1)-(2) is

$$\begin{cases}\dfrac{d\varphi(t)}{dt} &= A\varphi(t) + F(\varphi(t)),\\ \varphi(0) &= (\varphi_S,\varphi_I,\varphi_L)^T.\end{cases} \tag{3}$$

Let us make the following technical assumption:

Assumption 1. $\beta \in L^\infty_+[(0,a^+)\times(0,a^+)]$; $f,\ d_1,\ d_2,\ \gamma,\ \Lambda \in L^\infty_+(0,a^+)$; $\mu \in L^1_{loc}(0,a^+)$.

Lemma 1. *On assumption 1, the operators A and F are well defined.*

3. Main results

Lemma 2. *The operator A is generator of a C_0-semi-groupe of linear bounded operators $\{T(t)\}_{t\geq 0}$ such that*

$$\begin{aligned} T(t)\varphi(a) &= \begin{cases}\varphi(a-t) & si \quad a-t\geq 0\\ C(t-a) & si \quad a-t\leq 0\end{cases} && for\ \ t\leq a^+,\\ T(t)\varphi(a) &= 0_{\mathbb{R}^3} && for\ \ t>a^+,\end{aligned}$$

where $C(t) = (C_1(t), C_2(t), 0)$ is the unique solution of the following Volterra integral equation

$$C(t) = G(t) + \Phi(t,C),$$

with

$$G(t) = \left(\int_t^{a^+} f(s)(\varphi_1(s-t)+(1-p)\varphi_2(s-t)+\varphi_3(s-t))ds\ ;\ p\int_t^{a^+} f(s)\varphi_2(s-t)ds\ ;\ 0\right),$$

$$\Phi(t,C) = \left(\int_0^{t} f(s)(C_1(t-s)+(1-p)C_2(t-s))ds\ ;\ p\int_0^{t} f(s)C_2(t-s)ds\ ;\ 0\right).$$

Proposition 1. *[4] The domain $D(A)$ of operator A is dense in X and A is a closed operator.*

Let us note $X_+ = (L^1_+(0,a^+))^3$ the nonnegative cone of X. From the definition of $T(t)$ the following proposition holds.

Proposition 2. *The space X_+ is positively invariant by $\{T(t)\}_{t\geqslant 0}$.*

Proposition 3. *The trajectories of the system (1)-(2) are forward bounded.*

Lemma 3. *On assumption 1, the non linear term F is continuous and locally Lipschitz.*

We can now prove the existence and uniqueness of the solution of (3).

Theorem 1. *For any initial condition on X, the semi-linear problem (3) has a unique mild solution on $[0, +\infty)$.*

Let us note $l(a) = exp\left(-\int_0^a \mu(s)ds\right)$ the average lifetime of individuals at age a.

Assumption 2. *The functions f and l satisfy $\int_0^{a^+} f(a)l(a)da < 1$.*

Proposition 4. *On assumption 2, the system (1)-(2) has a unique Disease Free Equilibrium (DFE), $\varphi_0 = (S_0, 0_{L^1}, 0_{L^1})$, where S_0 is given by*

$$\begin{cases} S_0(0) = \dfrac{1}{1 - \int_0^{a^+} f(a)l(a)da} \displaystyle\int_0^{a^+} f(a)l(a) \left(\int_0^a \frac{\Lambda(s)}{l(s)} ds \right) da, \\ S_0(a) = l(a) \left[S_0(0) + \displaystyle\int_0^a \frac{\Lambda(s)}{l(s)} ds \right] \ for\ 0 \leq a \leq a^+. \end{cases} \tag{4}$$

In the following, we suppose that lost of sight individuals do not go back almost in a health center. Hence, the following assumption holds:

Assumption 3. $\gamma(a) \approx 0$ *for almost* $a \in (0, a^+)$.

The *basic reproduction ratio* R_0 is given by $R_0 := r(H_0)$; the spectral radius of the operator H_0 defined by $H_0(\psi)(a) = \int_0^{a^+} \chi(a, s)\psi(s)ds$, where

$$\begin{aligned} \chi(a,s) &= \frac{S_0(s)}{l(s)} \int_s^{a^+} \beta(a,\eta) \left(\chi_{21}(\eta, s) + \chi_{31}(\eta, s)\right) d\eta \\ &\quad + \frac{p\chi_4(s)}{\Delta(0)} \int_0^{a^+} \beta(a,\sigma)(A_{22}(\sigma) + A_{32}(\sigma))d\sigma \end{aligned}$$

and

$$\chi_4(a) = \left[\frac{S_0(a)}{l(a)} \int_0^{a^+} f(\sigma)l(\sigma)d\sigma - S_0(0) \right] \int_a^{a^+} f(s) \left[\chi_{21}(s,a) + \chi_{31}(s,a)\right] ds,$$

$$\Delta(0) = \left(1 - \int_0^{a^+} f(\sigma)l(\sigma)d\sigma\right) \left(p \int_0^{a^+} f(\sigma)[A_{22}(\sigma) + A_{32}(\sigma)]d\sigma - 1\right),$$

$$\chi_{21}(a,s) = l(a)\frac{\Gamma_1(a)}{\Gamma_1(s)} \quad ; \quad \chi_{31}(a,s) = l(a) \int_s^a \frac{\Gamma_2(a)\Gamma_1(\eta)}{\Gamma_2(\eta)\Gamma_1(s)} r(\eta)\phi(\eta)d\eta,$$

$$A_{22}(a) = l(a)\Gamma_1(a) \quad ; \quad A_{32}(a) = l(a)\Gamma_2(a) \int_0^a \frac{\Gamma_1(s)}{\Gamma_2(s)} r(s)\phi(s)ds,$$

$$\Gamma_1(a) = \exp\left(-\int_0^a (d_1(s) + r(s)\phi(s))ds\right) \quad ; \quad \Gamma_2(a) = \exp\left(-\int_0^a (d_2(s) + \gamma(s))ds\right).$$

As an important case, we here briefly consider the proposition mixing assumption, that is, the transmission rate β can be written as $\beta(a, s) = \beta_1(a)\beta_2(s)$ [2]. In this case we can define the basic reproduction R_0 explicitly by $R_0 := r(H_0) = \int_0^{a^+} \chi(s, s)ds$. Here we adopt the following technical assumption:

Assumption 4. *The transmission coefficient β satisfies the following:*

1) $\beta \in L^1_+(\mathbb{R}\times\mathbb{R})$ such that $\beta(a,s)=0$ for all $(a,s)\notin[o,a^+]\times[0,a^+]$.

2) $\lim\limits_{h\to 0}\int_{-\infty}^{+\infty}|\beta(a+h,\xi)-\beta(a,\xi)|da=0$ for all $\xi\in\mathbb{R}$.

3) There exists a nonnegative function ε such that $\varepsilon(s)>0$ for $s\in(0,a^+)$ and $\beta(a,s)\geqslant\varepsilon(s)$ for all $(a,s)\in(0,a^+)^2$.

Under assumption 4 we have the following results:

Proposition 5. *1) If $R_0\leq 1$, the DFE defined by (4) is the unique equilibrium of the system (1)-(2).*

2) If $R_0>1$, in addition to the DFE, the system (1)-(2) has at least one endemic equilibrium (S^*,I^*,L^*).

For the stability of the endemic equilibrium (S^*,I^*,L^*), let us assume that:

Assumption 5. *$\int_0^{a^+}(d_1(\sigma)+r(\sigma)\phi(\sigma))d\sigma\leq\exp\left(-\int_0^{a^+}\lambda^*(\sigma)d\sigma\right)$; where $\lambda^*(\sigma)=\int_0^{a^+}\beta(\sigma,\eta)(I^*(\eta)+L^*(\eta))d\eta$.*

Therefore, we have the following local stability result of our model:

Proposition 6. *1) If $R_0=r(H_0)<1$, the unique equilibrium (DFE) of the system (1)-(2) is locally asymptotically stable.*

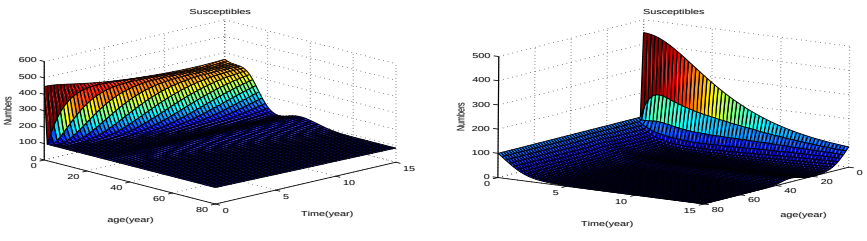
2) If $R_0=r(H_0)>1$, the DFE is instable.

3) If $R_0=r(H_0)>1$, the system (1)-(2) (in addition to the DFE) has at least one endemic equilibrium (EE). If more $r(V_0^*)<1$ and assumption 5 holds, the EE is locally asymptotically stable.

Where V_0^* is a positive linear operator.

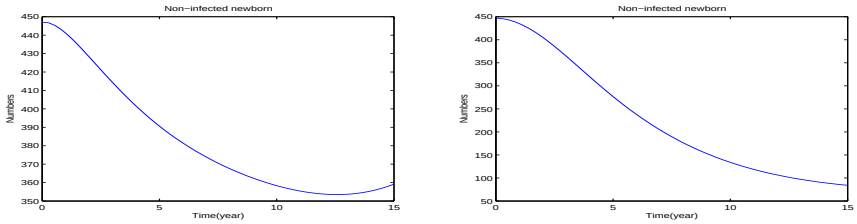
4. Numerical simulations

We adopt a numerical finite difference scheme. For the numerical simulations, the parameters of our system are arbitrarily chosen.



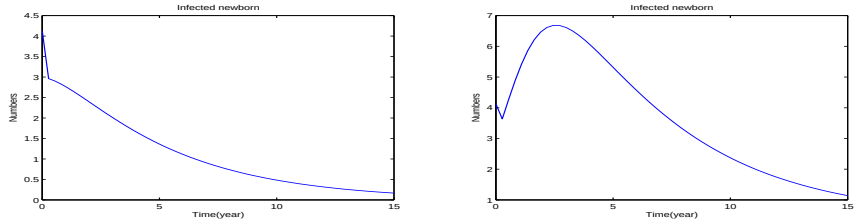
(a) (b)

Figure 2: (2a) Distribution of Susceptible individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0<1$. (2b) Distribution of Susceptible individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0>1$.



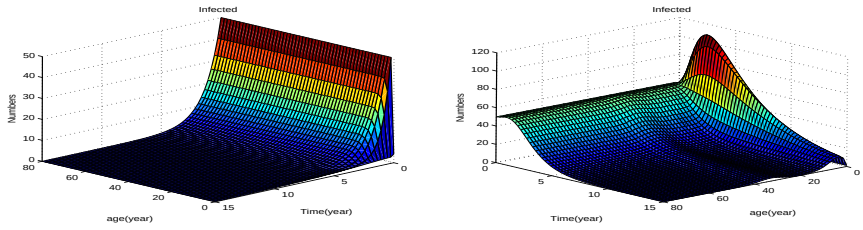
(a) (b)

Figure 3: (3a) Dynamical of Uninfected newborns individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 < 1$. (3b) Dynamical of Uninfected newborns individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 > 1$.



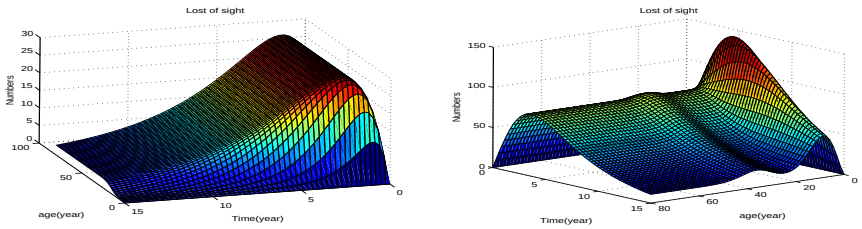
(a) (b)

Figure 4: (4a) Dynamical of Infected newborns individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 < 1$. (4b) Dynamical of Infected newborns individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 > 1$.



(a) (b)

Figure 5: (5a) Dynamical of Infected individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 < 1$. (5b) Dynamical of Infected individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 > 1$.



(a) (b)

Figure 6: (6a) Distribution of Lost of sight individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 < 1$. (6b) Distribution of Lost of sight individuals of Eq.(1)-(2) for arbitrarily chosen parameters such that $R_0 > 1$.

5. Conclusion

In this paper, we consider a mathematical model for the spread of a directly transmitted infections disease in an age-structured population with demographics process. The disease can be transmitted not only horizontally but also vertically from adult individuals to their children. The dynamical system is formulated with boundary conditions.

We have described the semigroup approach to the time evolution problem of the abstract epidemic system. Next we have calculated the basic reproduction ratio and proved that the disease-free steady state is locally asymptotically stable if $R_0 < 1$, and at least one endemic steady state exists if the basic reproduction ratio R_0 is greater than the unity. Moreover, we have shown that the endemic steady state is forwardly bifurcating from the disease-free steady state at $R_0 = 1$. Finally we have shown sufficient conditions which guarantee the local stability of the endemic steady state. Roughly speaking, the endemic steady state is locally asymptotically stable if it corresponds to a very small force of infection.

However the global stability of the model still an interesting open problem. Moreover, biologically appropriate assumptions for the unique existence of an endemic steady state is also not yet know.

6. References

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