Solutions to a Riemann problem at a junction of two reaches

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RÉSUMÉ. Nous étudions dans ce papier un problème de Riemann au niveau de la jonction de deux biefs. Les équations de Saint-Venant sont considérées dans chaque bief et des conditions particulières au niveau de la jonction. L'écoulement dans le canal est sous-critique. Nous avons montré, à l'aide d'une condition, que le problème de Riemann admet une solution unique. Cependant la condition n'est pas toujours vérifiée, autrement dit, l'existence d'une solution de Riemann, pour le problème considéré, n'est pas toujours garantie.

ABSTRACT. In this paper, we study a Riemann problem at a junction of two reaches. We consider the 1D Saint-Venant equations in each reach combined with particular conditions at the junction of the two reaches. The flow in the canal system is assumed to be subcritical. We have shown under a certain condition that the Riemann problem has a unique solution. However such a condition is not always satisfied, meaning that there are states where no solution to the Riemann problem can be found.

MOTS-CLÉS : Problème de Riemann, Equations de Saint-Venant, Systèmes hyperboliques, Canal à ciel ouvert.

KEYWORDS : Riemann problem, Saint-Venant equations, hyperbolic systems, open canal network.

1. Introduction

The so-called Saint-Venant equations are nonlinear system of one dimensional partial differential equations established by *Barré de Saint-Venant* in [1]. They are composed by a mass and momentum balance laws. In water management problems, these equations are often used as a fondamental tool to describe the dynamics of canals and rivers.

These equations are used in different type of configuration, involving sometimes also different type of junctions where several canals are interconnected. Among the configurations, we have star and cascade networks modeled as junction of canals, see [12], [15], [11] and reference cited therein. Concerning the junctions, they are originally derived by engineers and physical reasons motivate different choices of conditions, see [11]-[14]. In the present paper, we consider a canal with two reaches interconnected at a junction and assumed that the flow is subcritical. We use the conservation of mass and the non equality of water levels at the junction. An example involving such conditions is a cascade network [11].

When it comes to solve mathematical models involving discontinuities such as junctions, Riemann solvers are often used in order to handle correctly these discontinuities. Indeed discontinuities occur even when solving nonlinear models without junctions. These solvers are also used by numerical methods such as finite volumes and finite elements. Riemann problems have been, for a longtime, considered in the litterature for different types of problems among which : shallow water equations [2]-[4], gas networks [5]-[6] and traffic flows [7]-[8].

In this paper, we are inspired by the approach of [3]. The canal system and junction conditions considered in the present paper are different to those of [3]. A Riemann problem is formulated at a junction of two reaches. The solution to the Riemann problem is obtained through intersections of curves in state space. A condition ensuring the existence of a unique solution and an algorithm for computing the solution are given. However, there are states where no solution to the Riemann problem can be found.

2. Governing equations

Consider the configuration depicted at Fig. 1. The canal has a prismatic section of unit width and is modeled by 1D Saint-Venant equations in each reach together with junction conditions at M. The reaches before and after the junction are indexed by 1 and 2, respectively. The parameters are : h_i the height of the fluid column (m), v_i the flow $(m \cdot s^{-1})$, L_i the length of the canal (m). The governing equations are

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x} f_i(u_i) = 0, \quad \text{where} \quad u_i = \begin{pmatrix} h_i \\ v_i \end{pmatrix} \quad \text{and} \quad f_i(u_i) = \begin{pmatrix} v_i h_i \\ \frac{v_i^2}{2} + g h_i \end{pmatrix}. \tag{1}$$

Here g is the gravity constant. The Jacobian matrix of f_i and its eigenvalues are

$$f_i^{'}(u_i) = \begin{pmatrix} v_i & h_i \\ g & v_i \end{pmatrix}, \qquad \lambda_{i1} = v_i - \sqrt{g h_i}, \qquad \lambda_{i2} = v_i + \sqrt{g h_i}, \tag{2}$$

We consider a subcritical flow, a flow for which the speed v_i of the fluid is smaller than the speed $\sqrt{g h_i}$ of the gravity waves :

$$v_i < \sqrt{g h_i}. \tag{3}$$

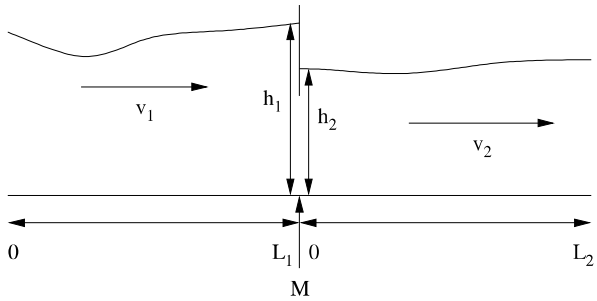


Figure 1. *The canal system.*

When a subcritical flow is considered, the eigenvalues $\lambda_{ij},\ j = 1, 2$ defined in (2) are of opposite sign and therefore there are two waves propagating in opposite directions. Note that critical cases are cases where states satisfy

$$v_i = \pm\sqrt{gh_i}. \tag{4}$$

The original shallow water equations are applied to both sides of the discontinuity but sometimes it is quite impossible to link these sides by using only the original equations. Instead, special conditions depending on the nature of the discontinuity are used. In our case, we have

$$h_1 v_1 = h_2 v_2, \tag{5}$$

$$h_1 > h_2. \tag{6}$$

These conditions express the conservation of mass and the non equality of water levels. A situation involving such conditions can be found in [11]. We consider, for simplicity, the following case of the water levels condition (6) :

$$h_1 = h_2 + \bar{h}, \tag{7}$$

where $\bar{h}$ is a constant given value.

In the sequel, we give rarefaction and shock curves from which the solution to the Riemann problem is obtained. More details can be found in [2], [9] and [10].

Rarefaction curves are computed using integral curves of right eigenvectors of the Jacobian of f_i. They determine the states u_i that can be connected to the initial state u_{i0} by physically correct rarefaction waves. They are

$$\mathcal{R}_1(u_{i0}) = \left\{ u_i \mid \quad v_i(h_i) = v_{i0} - 2\left(\sqrt{gh_i} - \sqrt{gh_{i0}}\right),\ \text{for } h_i < h_{i0} \right\}, \tag{8}$$

and

$$\mathcal{R}_2(u_{i0}) = \left\{ u_i \mid \quad v_i(h_i) = v_{i0} + 2\left(\sqrt{gh_i} - \sqrt{gh_{i0}}\right),\ \text{for } h_i < h_{i0} \right\}, \tag{9}$$

with initial condition $u_{i0} = u_i(x, 0) = (h_{i0}, v_{i0})$.

The Rankine-Hugoniot jump condition is used to determine shock curves. These shock curves determine the set of all states u_i that can be connected to the initial state u_{i0} by physically correct shock waves. They are

$$\mathcal{S}_1(u_{i0}) = \left\{ u_i \mid \quad v_i(h_i) = v_{i0} - \sqrt{\frac{2g}{h_i + h_{i0}}}(h_i - h_{i0}),\ \text{for } h_i > h_{i0} \right\}, \tag{10}$$

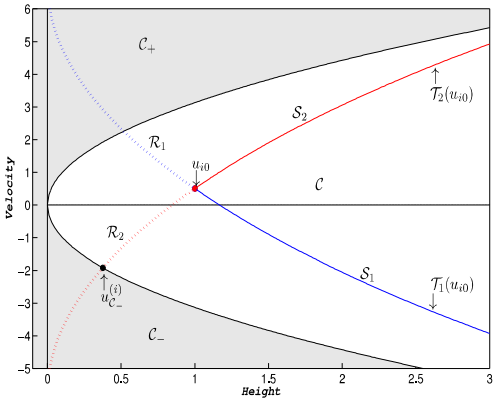


Figure 2. *Wave curves of state u_{i0}, $\mathcal{C}$ is a the subcritical region while $\mathcal{C}_-$ and $\mathcal{C}_+$ are non-subcritical regions.*

and

$$\mathcal{S}_2(u_{i0}) = \left\{ u_i \mid \quad v_i(h_i) = v_{i0} + \sqrt{\frac{2g}{h_i + h_{i0}}}(h_i - h_{i0}), \text{ for } h_i > h_{i0} \right\}. \tag{11}$$

Any given initial state u_{i0} is connected to the left and to the right by physically correct waves (rarefactions or shocks). Let consider the wave curves of first and second family defined as

$$\mathcal{T}_i(u_{i0}) = \mathcal{R}_i(u_{i0}) \cup \mathcal{S}_i(u_{i0}). \tag{12}$$

Curves of rarefaction and of shock are shown in Figure 2. The areas $\mathcal{C}_-$ and $\mathcal{C}_+$ (gray areas in Fig.2) contain states safisfying $v_i \leq -\sqrt{gh_i}$ and $v_i \geq \sqrt{gh_i}$, respectively. These are regions where the subcritical condition (3) is not satisfied. Therefore, any states in these regions will not be considered as an admissible states and thus will be rejected. Only states that are in region $\mathcal{C}$ (white area in Fig.2) are accepted because they satisfy the subcritical condition. The state $u_{\mathcal{C}_-}^{(i)}$ is the intersection between the wave curve $\mathcal{T}_2$ and the critical curve $\{v_i = -\sqrt{gh_i}\}$, and is, following the ideas of Lemma 3.3 in [3], given as follows

$$h_{\mathcal{C}_-}^{(i)} = \frac{1}{g}\left(-\frac{1}{3}v_{i0} + \frac{2}{3}\sqrt{gh_{i0}}\right)^2 \quad \text{and} \quad v_{\mathcal{C}_-}^{(i)} = -\sqrt{gh_{\mathcal{C}_-}^{(i)}} = \frac{1}{3}v_{i0} - \frac{2}{3}\sqrt{gh_{i0}}. \tag{13}$$

3. The Riemann problem

In order to state the Riemann problem, we rewrite (1) in the following form involving the junction M :

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{\partial}{\partial x} f_1(u_1) = 0, & x < M, \\ \dfrac{\partial u_2}{\partial t} + \dfrac{\partial}{\partial x} f_2(u_2) = 0, & x > M. \end{cases} \tag{14}$$

Recall that at the junction M, we have :

$$h_1 v_1 = h_2 v_2, \tag{15}$$

$$h_1 = h_2 + \bar{h}. \tag{16}$$

Then, the Riemann problem at the junction M is given by (14)-(16) together with the initial condition as Riemann data :

$$u_1(0,x) = u_{10}, \quad x < M, \qquad u_2(0,x) = u_{20}, \quad x > M. \tag{17}$$

The solution to this Riemann problem consists of two states separated by the intermediate states u_{1*} and u_{2*} satisfying (15)-(16). Furthermore, u_{1*} and u_{2*} are connected to the Riemann data through physically correct wave curve, i.e

$$u_{1*} \in \mathcal{T}_1(u_{10}) \quad \text{and} \quad u_{2*} \in \mathcal{T}_2(u_{20}) \tag{18}$$

where $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined in (12). We rewrite the mass conservation condition by using (16) in (15) to get

$$v_1 = \frac{h_2}{h_2 + \bar{h}} v_2. \tag{19}$$

It is straightforward, from (19), to show that if u_2 is subcritical, the state u_1 is also subcritical.

3.1. The junction curve

Solving the Riemann problem is reduced to looking for intersections between curves : wave and junction curves. The wave curves are already known (see (12)) and the junction curve comes from the compatibility conditions (7) and (15) together with wave curves. Thanks to (16) and (19), when u_2 follows the wave curve $\mathcal{T}_2(u_{20})$, the junction curve, denoted by $\mathcal{J}(u_{20})$ is given by :

$$\mathcal{J}(u_{20}) = \left\{ u \,|\quad h = h_2 + \bar{h}, \quad v = \frac{h_2}{h_2 + \bar{h}} v_2, \quad (h_2, v_2) \in \mathcal{T}_2(u_{20}) \right\}. \tag{20}$$

Let us recall from (9) and (11) that $(h_2, v_2) \in \mathcal{T}_2(u_{20})$ implies

$$v_2(h_2) = \begin{cases} v_{20} + 2\left(\sqrt{gh_2} - \sqrt{gh_{20}}\right), & \text{if} \quad h_2 < h_{20}, \quad (a) \\ v_{20} + \sqrt{\dfrac{2g}{h_2 + h_{20}}}(h_2 - h_{20}), & \text{if} \quad h_2 > h_{20}. \quad (b) \end{cases} \tag{21}$$

Therefore, the junction curve gives states (h, v) satisfying

$$h(h_2) = h_2 + \bar{h}, \qquad v(h_2) = \frac{h_2}{h_2 + \bar{h}} v_2(h_2), \qquad h_2 > 0. \tag{22}$$

In order to give a description of $\mathcal{J}(u_{20})$, let us consider the following result :

Lemma 1 *Let assume that u_2 is along the shock portion of the wave curve of the second family, i.e $(h_2, v_2(h_2))$ satisfies (21.b). Therefore, if the subcritical condition is satisfied, one has*

$$v^{'}(h_2) > 0, \qquad \forall\, h_2 > h_{20}. \tag{23}$$

The proof of Lemma 1 and that of next Lemma giving the description of the junction curve are given in the Appendix.

Lemma 2 *The junction curve is given by a function which is decreasing in* $]\bar{h}, h_{min}]$ *and increasing in* $[h_{min}, +\infty[$ *where* h_{min}

$$h_{min} = \bar{h} + h_{2,min}, \tag{24}$$

where $h_{2,min}$ *is the minimum of the function* $v(h_2)$ *and is given by*

$$h_{2,min} = \frac{1}{g}\left[\left(-\frac{q}{2} + \frac{1}{2}\sqrt{\frac{-\Delta}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \frac{1}{2}\sqrt{\frac{-\Delta}{27}}\right)^{\frac{1}{3}}\right]^2. \tag{25}$$

Here,

$$q = g\bar{h}(v_{20} - 2\sqrt{gh_{20}}) < 0, \qquad \Delta = -\left(27q^2 + 4(3g\bar{h})^3\right) < 0. \tag{26}$$

The junction curve decreases from

$$\lim_{h_2 \to 0^+} v(h_2) = 0 \quad \textit{to} \quad v_{min} = v(h_{2,min}) \tag{27}$$

and increases from

$$v_{min} = v(h_{2,min}) \quad \textit{to} \quad \lim_{h_2 \to +\infty} v(h_2) = +\infty. \tag{28}$$

Example 1 Let us consider the following parameters :

$$u_{10} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \qquad u_{20} = \begin{pmatrix} 1.6 \\ 1.2 \end{pmatrix}, \qquad \bar{h} = 0.5. \tag{29}$$

Fig.3(a) shows wave curves $\mathcal{T}_1(u_{10})$ and $\mathcal{T}_2(u_{20})$, the junction curve $\mathcal{J}(u_{20})$ and u_{min} the maximum state on the junction curve, which is given in Lemma 2.

3.2. Solution to the Riemann problem

Let us recall that we are solving the Riemann problem (14)-(17). The intermediate states in the solutions to the Riemann problem are determined thanks to the intersections of $\mathcal{J}(u_{20})$ and $\mathcal{T}_1(u_{10})$ (see Fig.3). These intersections solve the following equation

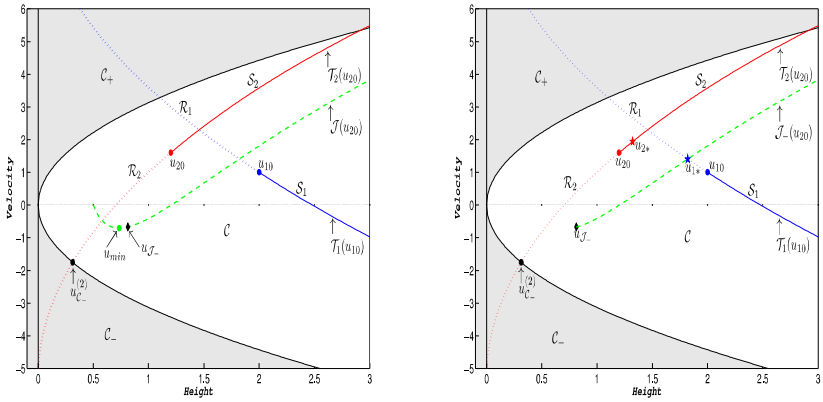
$$v_1(h_2 + \bar{h}) - \frac{h_2}{h_2 + \bar{h}} v_2(h_2) = 0, \tag{30}$$

where v_2 is given in (21) and v_1 is, from (8) and (10), defined as

$$v_1(h_1) = \begin{cases} v_{10} - 2\left(\sqrt{gh_1} - \sqrt{gh_{10}}\right), & \text{if} \quad h_1 < h_{10}, \quad (a) \\ v_{10} - \sqrt{\dfrac{2g}{h_1 + h_{10}}}(h_1 - h_{10}), & \text{if} \quad h_1 > h_{10}. \quad (b) \end{cases} \tag{31}$$

Note the wave curve of second family $\mathcal{T}_2(u_{20})$ contains both subcritical and non subcritical states since it crosses the critical curve $\{v_1 = \sqrt{gh_1}\}$. The crossing point is $u^{(2)}_{\mathcal{C}_-}$, see (13). Let us denote by $u_{\mathcal{J}_-}$ the corresponding state of $u^{(2)}_{\mathcal{C}_-}$ along $\mathcal{J}(u_{20})$. Therefore

$$h_{\mathcal{J}_-} = h^{(2)}_{\mathcal{C}_-} + \bar{h}, \qquad v_{\mathcal{J}_-} = \frac{h^{(2)}_{\mathcal{C}_-}}{h^{(2)}_{\mathcal{C}_-} + \bar{h}} v_2(h^{(2)}_{\mathcal{C}_-}). \tag{32}$$



(a) (b)

Figure 3. *(a) Wave and junction curves, maximum state u_{min} of $\mathcal{J}(u_{20})$ and $u_{\mathcal{J}_-}$ the corresponding state of $u^{(2)}_{\mathcal{C}_-}$ along $\mathcal{J}(u_{20})$. (b) The portion of $\mathcal{J}_-(u_{20})$ giving subcritical solution and the intermediate states u_{1*} and u_{2*} in the Riemann solution.*

The state $u_{\mathcal{J}_-}$ is shown in Fig.3 for data given in Example 1. States on the junction curve coming before $u_{\mathcal{C}_+}$ correspond to states on the wave curve $\mathcal{T}_1(u_{10})$ that are not subcritical. Therefore those states are not considered as valid states in the solution of the Riemann problem. The possible Riemann solutions come only from states after $u_{\mathcal{J}_-}$ and their corresponding states along $\mathcal{T}_2(u_{20})$. Denote this part of the junction curve by

$$\mathcal{J}_-(u_{20}) = \left\{ u \,|\quad h = h_2 + \bar{h}, \quad v = \frac{h_2}{h_2 + \bar{h}} v_2, \quad (h_2, v_2) \in \mathcal{T}_2(u_{20}), \quad h_2 > h^{(2)}_{\mathcal{C}_-} \right\}. \quad (33)$$

The curve $\mathcal{J}_-(u_{20})$ and the intermediate states u_{1*} and u_{2*} in Riemann solution are shown in Fig.3(b) for the same data as those of Example 1. In this case the u_{1*} is connected to u_{10} through a physically correct rarefaction wave and u_{2*} is connected to u_{20} through a physically correct shock wave, i.e, $(u_{1*}, u_{2*}) \in \big(\mathcal{R}_1(u_{10}), \mathcal{S}_2(u_{20})\big)$.

The following Theorem states a condition wich ensures the existence of a solution to the Riemann problem (14)-(17).

Theorem 1 *The Riemann problem (14)-(17) has a unique solution if and only if*

$$v_1(h_{\mathcal{J}_-}) > v_{\mathcal{J}_-}, \quad (34)$$

where v_1 and $(h_{\mathcal{J}_-}, v_{\mathcal{J}_-})$ are given by (31) and (32), respectively. In this case, the intermediate states in the Riemann solution, are

$$u_{1*} = \begin{pmatrix} h_{1*} \\ v_{1*} \end{pmatrix} = \begin{pmatrix} h_{2s} + \bar{h} \\ v_1(h_{2s} + \bar{h}) \end{pmatrix}, \; u_{2*} = \begin{pmatrix} h_{2*} \\ v_{2*} \end{pmatrix} = \begin{pmatrix} h_{2s} \\ v_2(h_{2s}) \end{pmatrix}, \quad (35)$$

where h_{2s} is a solution of

$$g(h_2) \equiv v_1(h_2 + \bar{h}) - \frac{h_2}{h_2 + \bar{h}} v_2(h_2) = 0, \quad h_2 > h^{(2)}_{\mathcal{C}_-}. \quad (36)$$

The proof of Theorem 1 is given in the Appendix.

Remark 1 *Note that by using (31) and Lemma 2, condition (34) can be checked computationally. The condition relates the initial states u_{10} and u_{20}. After chosing the initial state u_{20}, it is always possible to choose the velocity component v_{10} of u_{10} in order to satisfy condition (34).*

4. Bibliographie

[1] B. DE SAINT-VENANT , « Théorie du mouvement non-permanent des eaux avec application aux crues des rivières et à l'introduction des marées dans leur lit, », *Comptes Rendus Academie des Sciences,* vol. 73, 148 – 154, 237 – 240, 1871.

[2] A. MARIGO, « Entropic solutions for irrigation networks », *SIAM J. Appl. Math.*, vol. 70, n° 5, 1711-1735, 2010.

[3] M. S. GOUDIABY, G. KREISS, « A Riemann problem at a junction of open canals », *J. Hyp. Diff. Eq.*, vol. 10, n° 3, 431-460, 2013.

[4] R. BERNETTI, V. A. TITAREV, E. F. TORO, « Exact solution of the Riemann problem for shallow water equations with discontinuous bottom geometry », *Journal of Computational Physics*, vol. 227, 3212-3243, 2008.

[5] M. K. BANDA, M. HERTY, A. KLAR, « Coupling conditions for gas networks governed by the isothermal Euler equations », *Netw. Heterog. Media*, vol. 1, n° 2, 295-314, 2006.

[6] R. M COLOMBO, M. GARAVELLO, « A well posed Riemann Problem for the p-System at a junction », *Netw. Heterog. Media*, vol. 1, n° 3, 495-511, 2006.

[7] R. M COLOMBO, P. GOATIN, M. D. ROSINI, « On the Modeling and Management of Traffic », *ESAIM : Math. Model. Numer. Anal.*, vol. 45, 853-872, 2011.

[8] M. GARAVELLO, B. PICCOLI, « Traffic flow on a road network using the Aw-Rascle model », *Comm. Partial Differential Equations*, vol. 31, 243-275, 2006.

[9] R. J. LEVEQUE, « Finite Volume Methods for Hyperbolic Problems », *Cambridge Texts in Applied Mathematics,* USA, 2002.

[10] E. F. TORO, « Riemann Solvers and Numerical Methods for Fluid Dynamics : A Practical Introduction », *Springer-Verlag, Berlin*, 1999.

[11] J. DE HALLEUX, C. PRIEUR, J-M. CORON, B. D'ANDRÉA-NOVEL, G. BASTIN, « Boundary feedback control in networks of open channels », *Automatica*, vol. 39, n° 8, 1365-1376, 2003.

[12] G. LEUGERING, E. J. P. GEORG SCHMIDT « On the modelling and stabilization of flows in networks of open canal, » *SIAM J. Control Optim.*, vol. 41, n° 1, 164-180, 2002.

[13] J. A. CUNGE, F. M. HOLLY, A. VERWEY « Practical Aspects of Computational River Hydraulics, » *Pitman*, Boston, 1980.

[14] M. GUGAT, G. LEUGERING, E. J. P. GEORG SCHMIDT « Global controllability between steady supercritical flows in channel networks, » *Mathematical Methods in the Applied Sciences*, vol. 27, 781-802, 2004.

[15] M. S. GOUDIABY, A. SENE, G. KREISS « A delayed feedback control for network of open canals, » *Int. J. Dynam. Control*, vol. 1, n° 4, 316-329, 2013.