Fast inversion of triangular Toeplitz matrices based on trigonometric polynomial interpolation

Skander Belhaj*,** — Marwa Dridi**

* Université de Manouba
Institut Supérieur des Arts du Multimédia de la Manouba (ISAMM)
Campus Universitaire de Manouba - 2010 Tunis
Tunisie
skander_belhaj@yahoo.fr

** Université de Tunis El Manar
ENIT - LAMSIN
BP 37, 1002 Tunis
Tunisie
dridi_marwa23@yahoo.fr

RÉSUMÉ. En utilisant l'interpolation polynomiale trigonométrique, un algorithme numérique rapide et efficace pour le calcul de l'inverse d'une matrice triangulaire de Toeplitz avec des coefficients réels a été proposé dans la littérature. La complexité de l'algorithme est de deux transformées de Fourier rapides (FFT) et d'une transformée de cosinus rapide (DCT) de $2n$-vecteurs. Dans cet article, nous présentons un algorithme avec une complexité de deux transformées de Fourier rapide (FFT) de $2n$-vecteurs pour calculer l'inverse d'une matrice de Toeplitz triangulaire avec des nombres réels et/ou complexes. Une analyse théorique de la précision est également considéré. Des exemples numériques sont donnés pour illustrer l'efficacité de notre méthode.

ABSTRACT. Using trigonometric polynomial interpolation, a fast and effective numerical algorithm for computing the inverse of a triangular Toeplitz matrix with real numbers has been recently proposed. The complexity of the algorithm is two fast Fourier transforms (FFTs) and one fast cosine transform (DCT) of $2n$-vectors. In this paper, we present an algorithm with two fast Fourier transforms (FFTs) of $2n$-vectors for calculating the inverse of a triangular Toeplitz matrix with real and/or complex numbers. A theoretical accuracy analysis is also considered. Numerical examples are given to illustrate the effectiveness of our method.

MOTS-CLÉS : Interpolation polynomiale trigonométrique, matrice triangulaire de Toeplitz, Transformation de Fourier rapide

KEYWORDS : Trigonometric polynomial interpolation, Triangular Toeplitz matrix, Fast Fourier transforms

1. Introduction

Let lT_n be an n-by-n lower triangular Toeplitz matrix :

$$lT_n = \begin{pmatrix} t_0 & & & \\ t_1 & t_0 & & \\ \vdots & \ddots & \ddots & \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}, \tag{1}$$

where $(t_j)_{j=1,\ldots,n-1}$ are real and/or complex numbers. Problems related to compute the inverse of a nonsingular lower triangular Toeplitz matrix often appear in several fundamental problems of scientific computing, signal and image processing, etc [8]. To compute the inverse of a lower triangular Toeplitz matrix with real numbers, Lin, Ching and Ng [7] give an efficient numerical algorithm whose computational complexity is two fast Fourier transforms (FFTs) and one fast cosine transform (DCT) of $2n$-vectors. Other well-known algorithms, i.e. Bini's and Pan-Chen's algorithms for computing the inverse of a lower triangular Toeplitz matrix are given [4, 9].

In this paper, a new algorithm is developed for computing the inverse of a triangular Toeplitz matrix with real and/or complex numbers. The key issue of our method is to adopt the framework of approximate matrix inversion and employ techniques based on interpolation via trigonometric polynomials, following an idea proposed by Lin, Ching and Ng [7]. The complexity of our method for computing the inverse of a triangular Toeplitz matrix with real and/or complex numbers is two FFTs of $2n$-vectors. A theoretical accuracy analysis is also considered. Several numerical examples are given to illustrate the effectiveness and stability of the proposed algorithm with respect to the ones provided by the known algorithms.

The rest of this paper is organized as follows : In the next section, we give some classical results. In section 3, our algorithm is presented to compute the inverse of a triangular Toeplitz matrix with real and/or complex numbers. In Section 4, some numerical examples are introduced to show the performance of our algorithm. Finally, we make some concluding remarks in Section 5.

2. Some classical results

To make the paper self-contained we provide the following resume of Toeplitz matrices.

Definition 2.1 $T_n = [t_{ij}]_{i,j=0}^{n-1}$ is a Toeplitz matrix if $t_{ij} = t_{i+k,j+k}$ for all positive k (finite), that is, if all the entries of T_n are invariant in their shifts in the diagonal direction, so that the matrix T_n is completely defined by its first row and its first column.

Toeplitz matrix of size n is completely specified by $2n-1$ parameters, thus requiring less storage space than ordinary dense matrices. Moreover, many computations with Toeplitz matrices can be performed faster ; this is the case, for instance, for the sum and the product by a scalar. Now, we list some definitions and more advanced results.

Proposition 2.1 [1] The multiplication of a Toeplitz matrix of size n by a vector can be reduced to multiplication of two polynomials of degree at most $2n$ and performed with a computational cost of $O(n \log n)$.

Definition 2.2 $Z_f(\mathbf{c}) = Z_{f,m,n}(\mathbf{c}) = [z_{i,j}]$, for a vector $\mathbf{c} = [c_0, ..., c_{m-1}]^T$ and for a scalar $f \neq 0$, is an f-circulant $n \times n$ matrix if $z_{i,j} = c_{i-j}$ for $i \geq j$; $z_{i,j} = f c_{n+i-j}$ for $i < j$.

Definition 2.3 $lT_n = lT_n(t) = Z_0(t)$ denotes the lower triangular Toeplitz matrix with the first column t, that is, $lT_n = \sum_{i=0}^{n-1} t_i Z^i$ where $t = (t_0, ..., t_{n-1})^T$ and Z is the down-shift matrix filled with zeros, except for its first lower subdiagonal filled with ones.

Lemma 2.1 The products and inverses of f-circulant $n \times n$ matrices are f-circulant $n \times n$ matrices.

Lemma 2.2 For a lower triangular Toeplitz matrix lT_n, we define the polynomial :

$$p_n(z) = \sum_{k=0}^{n-1} t_i z^k = t_0 + t_1 z + ... + t_{n-1} z^{n-1} \tag{2}$$

Let the Maclaurin series of $p_n^{-1}(z)$ be given by

$$p_n^{-1}(z) = \sum_{k=0}^{\infty} v_k z^k, \tag{3}$$

then

$$lT_n^{-1} = \begin{pmatrix} v_0 & & & \\ v_1 & v_0 & & \\ \vdots & \ddots & \ddots & \\ v_{n-1} & \cdots & v_1 & v_0 \end{pmatrix} \tag{4}$$

Thus in order to obtain lT_n^{-1}, we only need to compute the coefficients v_k for $k = 0, 1, ..., n-1$.

Lemma 2.3 Replacing z in (2) and (3) by ρz we get $p_{n,\rho}(z) = p_n(\rho z) = \sum_{k=0}^{n-1} \left(t_k \rho^k\right) z^k$ and $p_{n,\rho}^{-1}(z) = p_n^{-1}(\rho z) = \sum_{k=0}^{\infty} \left(v_k \rho^k\right) z^k$. Equivalently, we have

$$\begin{pmatrix} t_0 & & & \\ \rho t_1 & t_0 & & \\ \vdots & \ddots & \ddots & \\ \rho^{n-1} t_{n-1} & \cdots & \rho t_1 & t_0 \end{pmatrix}^{-1} = \begin{pmatrix} v_0 & & & \\ \rho v_1 & v_0 & & \\ \vdots & \ddots & \ddots & \\ \rho^{n-1} v_{n-1} & \cdots & \rho v_1 & v_0 \end{pmatrix}$$

We note that we can choose $\rho \in (0,1)$ such that $\sum_{k=0}^{\infty} \left|v_k \rho^k\right| < \infty$.

Lemma 2.4 The inverse of the leading principal sub-matrix $lT_n(1:m, 1:m)$ is equal to the leading principal sub-matrix $lT_n^{-1}(1:m, 1:m)$ for $1 \le m \le n$.

3. Main results

Without loss of generality, suppose that $z = e^{-ix}$, where $i^2 = -1$ and x is a real variable, we deduce that $p_n(e^{-ix})$ is a trigonometric polynomial. One possible way to obtain v_k is to compute the Fourier coefficients of $1/p_n(e^{-ix})$: $v_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p_n(e^{-ix})} e^{ikx} dx$ for $k = 0, 1 \ldots, n-1$. It is hard to compute v_k explicitly since p_n^{-1} is generally unknown. This is why we propose an approximate way to conclude p_n^{-1} as defined in (3). Let $p_\rho(\theta) = p_{n,\rho}(e^{-i\theta}) = \sum_{k=0}^{n-1}(t_k\rho^k)e^{-ik\theta} = p_\rho^{(r)}(\theta) + ip_\rho^{(i)}(\theta)$ where $p_\rho^{(r)}(\theta)$ and $p_\rho^{(i)}(\theta)$ are the real and imaginary part of $p_\rho(\theta)$, respectively. Similarly, by using (3), we have $h_\rho(\theta) \equiv p_\rho^{-1}(\theta) = \sum_{k=0}^{\infty} x(v_k\rho^k)e^{-ik\theta}$ then

$$h_\rho(\theta) = \frac{p_\rho^{(r)}(\theta) - ip_\rho^{(i)}(\theta)}{(p_\rho^{(r)}(\theta))^2 + (p_\rho^{(i)}(\theta))^2}, \tag{5}$$

Noticeably, $h_\rho^{(r)}(\theta)$ and $h_\rho^{(i)}(\theta)$ are the real and imaginary part of $h_\rho(\theta)$ which are 2π-periodic even and odd function, respectively. To obtain approximate values $\widehat{v}_k$ for v_k, $k = 0, 1, \ldots, n-1$, we interpolate $h_\rho(\theta)$ by function in $\prod_{n-1}$, where $\prod_m$ denotes the set of all even trigonometric polynomials of degree $\le m$. We use the following equidistant points $\theta_k = \frac{2k}{n}\pi$, $k = 1, 2, \ldots, n$ as the interpolating knots to obtain efficiently $\widehat{v}_k$ by using FFTs and to approximate the original function accurately via the interpolating trigonometric polynomial. Let $\tau_{n-1}(\theta) = \sum_{k=0}^{n-1} f_k e^{-ik\theta}$ be the corresponding interpolating polynomial for $h_\rho(\theta)$. By using the interpolating condition $\tau_{n-1}(\theta_k) = h_\rho(\theta_k)$, $k = 1, \ldots, n$,

we have $\mathcal{F}(f_0, f_1, \ldots, f_{n-1}) = (h_\rho(\theta_1), h_\rho(\theta_2), \ldots, h_\rho(\theta_n))^t$ where $[\mathcal{F}]_{j,k} = e^{-2ikj/n}$. Note that $\mathcal{F}$ is the FFTs matrix and we see that if the values of $h_\rho(\theta)$, $k = 1, 2, \ldots, n$ are known, then f_k can be obtained by using one FFTs of n-vector. Finally, $\widehat{v}_k$ can obtained in $O(n)$ divisions by making use of $\widehat{v}_k = f_k \rho^{-k}$.

3.1. Algorithm and Complexity for computing the inverse of a triangular Toeplitz matrix

In the following, we give the algorithm for computing the inverse of a triangular Toeplitz matrix with real and/or complex numbers.

Algorithm 3.1

Step 0 : Based on Remark 3.1, choose $\rho \in (0, 1)$ and compute $\tilde{t_j} = \rho^j t_j$, for $j = 0, 1, \ldots, n-1$.

Step 1 : Compute $p_\rho(\theta_k) = \sum_{l=0}^{n-1} \tilde{t}_l e^{-il\theta_k}$ where $\theta_k = 2k\pi/2n$ for $k = 1, ..., 2n$.

Step 2 : Compute $h_k = \frac{p_\rho^{(r)}(\theta_k) - ip_\rho^{(i)}(\theta_k)}{(p_\rho^{(r)}(\theta_k))^2 + (p_\rho^{(i)}(\theta_k))^2}$, for $k = 1, ..., 2n$.

Step 3 : Solve $\mathcal{F}(f_0, f_1, \ldots, f_{2n-1})^t = (h_1, h_2, \ldots, h_{2n})^t$, where $[\mathcal{F}]_{j,k} = e^{-i\frac{2jk\pi}{2n}}$, $j, k = 1, 2, \ldots, 2n$. Compute $[\widehat{v}_k]_{k=0}^{n-1} = [f_k \rho^{-k}]_{k=0}^{n-1}$.

Complexity of Algorithm 3.1 In Step 1, the values of $p_\rho(\theta_k)$ for $k = 1, ..., 2n$ can be computed by one FFT$(2n)$. We make use of the FFT's matrix $[\mathcal{F}]_{j,k} = e^{-i\frac{2jk\pi}{2n}}$, $j, k = 1, 2, \ldots, 2n$. Therefore, we need about one FFT$(2n)$ for computing $\widehat{v}_k$, $k = 0, ..., n-1$ in Step 3.

Now, we compare costs among Bini's algorithm [4], revised Bini's algorithm [7], Lin-Chan-Ng's algorithm [7] and our algorithm (Algorithm 3.1) in Table 3.1.

Bini [4]	Lin-Chan-Ng [7]	revised Bini [7]	our algorithm
2 FFT(n)	2 FFT$(2n)$ + 1 DCT$(2n)$	2 FFT$(2n)$	2 FFT$(2n)$

Table 3.1. Comput. costs comparison for an $n \times n$ triangular Toeplitz matrix.

We prove that the computational cost of our algorithm for finding the inverse of a triangular Toeplitz matrix with real and/or complex numbers is less or equal than those of three well-known algorithms. Obviously, Bini's algorithm [4] requires about half cost

of the proposed method but this method is suitable when not requiring very high order of accuracy, such as in the Gauss-Seidel iteration for Toeplitz systems. In some applications, such as in [2], we need an approximation inversion as accurate as possible. In these cases, the Lin-Chan-Ng's algorithm, revised Bini's algorithm and our algorithm are more preferred.

3.2. Analysis of Algorithm 3.1 : Theoretical accuracy

Here we present a brief analysis of the theoretical accuracy of $[\widehat{v}_k]_{k=0}^{n-1}$ computed by Algorithm 3.1.

Theorem 3.1 Let the Maclaurin series of $p_n^{-1}(\rho z)$ be given by $\sum\limits_{j=0}^{\infty}(v_j\rho^j)z^j$ and $\rho \in (0,1)$ such that $\sum\limits_{j=0}^{\infty}|v_j\rho^j| < \infty$. Let $\tau_{n-1}(\theta) = \sum_{j=0}^{n-1} f_j e^{-ij\theta}$ be the interpolating polynomial for $h_\rho(\theta)$ with interpolating knots $\theta_k = \frac{2k}{n}\pi$, $k = 1, 2, \ldots, n$ and $\widehat{v}_j = f_j\rho^{-j}$. Then $|\widehat{v}_k - v_k| = O(\rho^n),\ k = 0, 1, \ldots, n-1$. More precisely,

$$\widehat{v}_k = v_k + \rho^n \sum_{j=1}^{\infty}(\rho^{(j-1)n} v_{k+jn}),\ k = 1, 2, \ldots, n-1. \tag{6}$$

Proof. Let $w = e^{-2i\pi/n}$. We have $f_k = \frac{1}{n}\sum_{j=0}^{n-1}(\sum_{l=0}^{\infty} v_l\rho^l w^{lj}) = \sum_{m=0}^{\infty} v_{k+mn}\rho^{k+mn}$. The last equality is a result of discrete orthogonality relation. Thus we have the following deducing formula : $\sum_{j=0}^{n-1} w^{-lj}w^{kj} = \sum_{j=0}^{n-1} w^{(k-l)j} = \frac{1-w^{(k-l)n}}{1-w^{k-l}}$, where

$$\frac{1-w^{(k-l)n}}{1-w^{k-l}} = \begin{cases} 0 \text{ if } k \neq l \pmod{n} \\ n \text{ if } k = l \pmod{n} \end{cases}$$

■

Remark 3.1 According to our preliminary numerical tests, we remark that $\rho = \exp(-9/n)$ and $\rho = (0.5 \times \exp(-11))^{1/n}$ are good choices for our method in real and complex cases, respectively.

To illustrate the results of Theorem 3.1, we plot in Fig. 3.1 the errors $(\log_{10}(|\hat{v} - v|))$ in finding the inverse of a triangular Toeplitz matrix with entries given by $t_k = \frac{1}{(k+1)^2},\ k = 0, 1 \ldots, 511$, $\rho = \exp(-9/n)$ and $\rho = 1$. Here v is the first column of the inverse of T_n obtained by the divide-and-conquer method [6] and $\hat{v}$ is the first column of the approximate inverse obtained by our method.

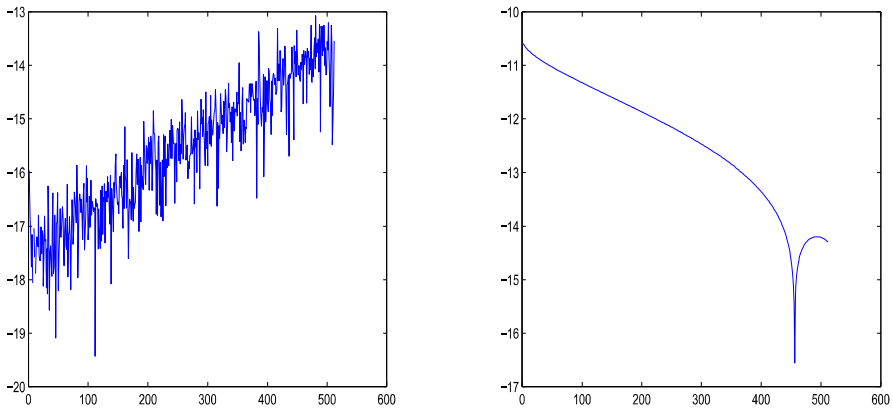

Fig. 3.1. $\log_{10}(|\hat{v} - v|)$ with $t_k = \frac{1}{(k+1)^2}$, $k = 0, 1, \ldots, 511$.
The left one is for $\rho = \exp(-9/n)$ and the right one is for $\rho = 1$.

It is clear that the numerical results coincide with the theoretical results.

4. Examples and numerical tests

In this section, we give the results of some numerical tests to illustrate the effectiveness of our algorithm. All examples were performed in MATLAB R2007a using double precision arithmetic.

Tables 4.1-4.2 give the behavior of the relative accuracy $\frac{\|\widehat{v}-v\|_1}{\|v\|_1}$ of Bini's algorithm (with real and/or complex numbers) [4], revised Bini algorithm (with real and/or complex numbers) [7], Lin-Chan-Ng's Algorithm (with real numbers solely) [7] and our algorithm (with real and/or complex numbers) for two different sequences of lower triangular Toeplitz matrices where $\widehat{v}$ is the first column of the approximate inverse and v is the column of the exact inverse computed by the divide-and-conquer method [6].

n	Bini	Interpolation	Revised Bini	Modified-Interpolation
512	2.5446×10^{-008}	5.4048×10^{-011}	1.3059×10^{-011}	2.3302×10^{-012}
1024	4.4955×10^{-008}	8.8015×10^{-011}	2.2790×10^{-011}	2.7815×10^{-012}
2048	5.2207×10^{-008}	1.0615×10^{-010}	3.3609×10^{-011}	3.7570×10^{-012}
4096	8.8215×10^{-008}	1.5081×10^{-010}	5.1190×10^{-011}	5.6812×10^{-012}

Table 4.1. $t_k = \frac{1}{(k+1)}$, $k = 0, 1, \ldots, n-1$.

n	Bini	Interpolation	Revised Bini	Modified-Interpolation
512	2.3745×10^{-008}	4.9944×10^{-011}	1.4104×10^{-011}	1.4950×10^{-012}
1024	3.7879×10^{-008}	8.6915×10^{-011}	2.1475×10^{-011}	2.2359×10^{-012}
2048	5.0758×10^{-008}	1.1980×10^{-010}	3.2501×10^{-011}	3.1058×10^{-012}
4096	8.8215×10^{-008}	1.5081×10^{-010}	5.1190×10^{-011}	5.6812×10^{-012}

Table 4.2. $t_k = \frac{1}{(k+1)^2}$, $k = 0, 1, \ldots, n-1$.

While the above sequences with real numbers (Tables 4.1-4.2) are very favorable for computing the inverse of a triangular Toeplitz matrix, they demonstrate that the proposed method (Algorithm 3.1) gives similar or best results. The following sequences with complex numbers (Tables 4.3-4.4) show the competitiveness of the proposed method.

n	Bini	Revised Bini	Modified-Interpolation
512	6.3364×10^{-009}	1.0184×10^{-010}	2.1983×10^{-011}
1024	1.1392×10^{-008}	9.8914×10^{-011}	1.6085×10^{-011}
2048	1.3366×10^{-008}	1.0350×10^{-010}	2.6875×10^{-011}
4096	2.0866×10^{-008}	1.0491×10^{-010}	7.0000×10^{-011}

Table 4.3. $t_0 = t_1 = 1 + i$, $t_k = 0$, $k = 2, 3 \ldots, n-1$.

n	Bini	Revised Bini	Modified-Interpolation
512	5.0319×10^{-008}	1.4823×10^{-010}	9.3484×10^{-011}
1024	1.0113×10^{-007}	1.8976×10^{-010}	1.8405×10^{-010}
2048	1.9431×10^{-007}	2.5356×10^{-010}	3.5098×10^{-010}
4096	4.5384×10^{-007}	4.3653×10^{-010}	7.1259×10^{-010}

Table 4.4. $t_k = \frac{1}{2} + i\left(\frac{1}{k+1}\right)$, $k = 0, 1 \ldots, n-1$.

5. Concluding remarks

In this paper, a numerical algorithm for computing the inverse of a triangular Toeplitz matrix with real and/or complex numbers are presented. We have showed that the computational cost of our algorithm for finding the inverse of a triangular Toeplitz matrix with real and/or complex numbers is less or equal than those of three well-known algorithms. A theoretical error analysis is also provided in [3].

6. Bibliographie

[1] J. Abdeljaoued, H. Lombardi : Méthodes matricielles. Introduction à la complexité Algébrique. Springer, collection "Mathématiques et applications" de la SMAI (2003).

[2] S. Belhaj, Computing the polynomial remainder sequence via Bézout matrices, J. Computational Applied Mathematics 250, 244-255 (2013).

[3] S. Belhaj and M. Dridi, A note on computing the inverse of a triangular Toeplitz matrix, Applied Mathematics and Computation 236, 512-523 (2014).

[4] D. Bini : Parallel solution of certain Toeplitz linear systems. SIAM J. Comput. 13, 268-276 (1984).

[5] D. Bini, V. Pan : Polynomial and Matrix computations. Vol.1, Fundamental Algorithms, Birkhauser (1994).

[6] D. Commenges, M. Monsion : Fast inversion of triangular Toeplitz matrices. IEEE Trans. Automat. Control AC-29, 250-251 (1984).

[7] Fu-Rong Lin, Wai-Ki Ching, M. K. Ng : Fast inversion of triangular Toeplitz matrices. Theoretical Computer Science 315, 511-523 (2004).

[8] V. Y. Pan : Structured Matrices and Polynomials. Unified Superfast Algorithms, Springer Verlag (2001).

[9] V. Y. Pan, Z. Q. Chen, Approximate real polynomial division via approximate inversion of real triangular Toeplitz matrices, Appl. Math. Letters 12, 1-2 (1999).