An inverse boundary value problem for image inpainting

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ABSTRACT. Image inpainting or disocclusion, is a classical problem in image processing. Which refers to the process of restoring a damaged image with missing information. Many variational models have appeared in the literature to solve this problem. These models give rise to partial differential equations for which Dirichlet boundary conditions are usually used. The basic idea of the algorithms that have been proposed is to fill-in damaged regions with available information from their surroundings. The aim of this work is to treat the case where this information is not available in a part of the boundary of the damaged region. The inpainting problem is formulated as an elliptic nonlinear Cauchy problem. Then, we give a Nash-game formulation of this Cauchy problem and we present different numerical experiments using the finite-element method for solving the image inpainting problem.

RÉSUMÉ. La désocclusion (inpainting) est un problème classique en analyse d'images qui consiste à retrouver des parties cachées ou endommagées d'une image. Dans la littérature, on trouve beaucoup de modèles variationnels pour résoudre ce problème. Ces modèles considèrent des équations aux dérivées partielles pour lesquelles la condition aux limites de type Dirichlet est généralement utilisée. L'idée de base dans ces algorithmes est de remplir les régions endommagées à partir des informations disponibles au voisinage de sa frontière. Le but de ce travail est de traiter le cas où cette information n'est pas disponible sur une partie de la frontière de la zone endommagée. Notre approche consiste à résoudre un problème de Cauchy non linéaire elliptique par une stratégie de jeu de Nash. Cette approche a donné lieu à un algorithme de reconstruction, qui a été suivi par une validation numérique effectuée sur plusieurs types d'images.

KEYWORDS : Image inpainting, Cauchy problem, Nash game.

MOTS-CLÉS : Désocclusion d'image, Problème de Cauchy, Jeu de Nash.

1. Introduction

In this work we address a Cauchy problem arising in image inpainting which consists in reconstructing lost or deteriorated parts of an image. Different techniques can be applied to solve this problem. For instance, partial differential equations (PDE) are widely used and are proven to be efficient (see e.g. [2]). Let Ω denote the entire image domain, the problem is to fill-in image information in the incomplete/damaged region $D \subset \Omega$ based upon the image information available outside D (i.e., in $\Omega \setminus D$). We assume that

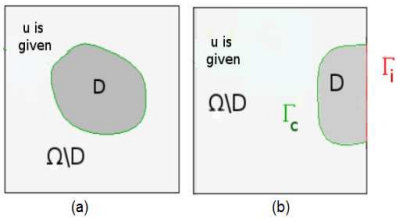

Figure 1. *Illustration of a typical inpainting problem.*

∂D is sufficiently smooth and composed of two connected components Γ_c and Γ_i. When the information is available near all the boundary $\partial D = \Gamma_i \cup \Gamma_c$ (see Fig. 1 (a)), it can be used as Dirichlet boundary conditions for the partial differential equation that propagates the information inside D such as in [2, 3]. The aim of this work is to treat the case where this information is not available in the part Γ_i (see Fig. 1 (b)). The inpainting problem is formulated as a Cauchy problem, which is linear in the case of smooth images and nonlinear for images containing edges. In both cases, we consider the following inverse boundary-value problem for the unknown image function u:

$$\begin{cases} \nabla\cdot[k(|\nabla u|^2)\nabla u] = 0, & \text{in } D, \\ u = f, & \text{on } \Gamma_c, \\ k(|\nabla u|^2)\nabla u \cdot n = \phi, & \text{on } \Gamma_c. \end{cases} \tag{1}$$

The Cauchy problem for elliptic equations still remains a challenge for modern analysis [4, 6, 7, 9]. It is generally an ill-posed problem in the sense of Hadamard which makes classical numerical methods usually inappropriate because they are unstable, so there is a need for carefully stabilized and dedicated computational methods for the numerical treatment. Regularization methods through reformulation of the Cauchy problem itself was introduced in [10, 11]. Others called iterative methods, for solving Cauchy problems for the Laplace equation, was introduced in [6, 7]. I. Ly and N. Tarkhanov in [11] give a new method, they rephrase the nonlinear Cauchy for first order as a variational problem and they have developed this method in [12] to solve Cauchy problems for the p-Laplace operator.

The novelty of this work has two folds. First, it consists in extending the work introduced in [13] for a nonlinear elliptic equation. The Cauchy problem is formulated as a two-player Nash game. The first player is given the known Dirichlet data and uses the Neumann condition prescribed over the inaccessible part of the boundary as strategy variable. The second player given the known Neumann data, and plays with the Dirichlet condition prescribed over the accessible boundary. The two players solve, in parallel, the associated boundary value problems. Second, the proposed approach is exploited in im-

age inpainting. The image information in the incomplete/damaged region is completed by the corresponding Cauchy solution.

2. The nonlinear direct problem

In this work, we consider the following function

$$k(s) = \frac{1}{\sqrt{s+\epsilon^2}} + \alpha, \tag{2}$$

where $\alpha > 0$ and $\epsilon > 0$ are small parameters. The function k is a regularized version of the diffusion function used in total variation (TV) [2] which consists of joining the same level lines on both sides of D and minimizing their lengths. We note that this function satisfing the following conditions.

$$\begin{cases} (a) & \alpha + \frac{1}{\epsilon} \geq k(s) \geq \alpha > 0, \ \forall s \in \mathbb{R}^+; \\ (b) & k'(s) \leq 0, \ \forall s \in \mathbb{R}^+; \\ (c) & k(s) + 2sk'(s) \geq \alpha > 0, \ \forall s \in \mathbb{R}^+. \end{cases} \tag{3}$$

We consider the following direct nonlinear boundary value problem:

$$\begin{cases} \nabla\cdot[k(|\nabla u|^2)\nabla u] = 0, & \text{in } D, \\ u = f, & \text{on } \Gamma_c, \\ k(|\nabla u|^2)\nabla u \cdot n = \varphi, & \text{on } \Gamma_i. \end{cases} \tag{4}$$

Assumptions (a), (b) and (c) are the naturel sufficient conditions to guarantee the solvability of (4). The existence and uniqueness of the solution to (4) is garanteed by the theory of monotone operators (see also [14, 15, 16]).

The linearized direct problem can be rewritten in the following form: Given an initial approximation u_0 of the solution u, we consider the sequence $(u_n)_{n\geq 1}$ where u_n is the solution of

$$\begin{cases} \nabla\cdot[k(|\nabla u_{n-1}|^2)\nabla u_n] = 0, & \text{in } D, \\ u_n = f, & \text{on } \Gamma_c, \\ k(|\nabla u_{n-1}|^2)\nabla u_n \cdot n = \varphi, & \text{on } \Gamma_i. \end{cases} \tag{5}$$

On each iteration, the linearization leads to a linear elliptic equation which has a unique solution $u_n \in H^1(D)$.

Theorem 2.1 *The solution $u_n \in H^1(D)$ of the linearized problem converges, in H^1-norm as $n \to \infty$, to the solution $u \in H^1(D)$ of the nonlinear problem (4).*

3. Existence and uniqueness of the Cauchy problem solution

The existence of a solution of our Cauchy problem (1) depends the compatibility of the Cauchy data f and ϕ. The data f and ϕ are said to be compatible if the Cauchy problem (1) admits a solution. We also say that f and ϕ are consistent for (1). Let us define $[H_{00}^{1/2}(\Gamma_c)]'$ as the dual space of $H_{00}^{1/2}(\Gamma_c)$, which consists of functions in $H^{1/2}(\Gamma_c)$, vanishing on Γ_i. Then we have:

Proposition 1 *For $f \in H^{1/2}(D)$, we define the set*

$$M = \{\phi \in [H^{1/2}_{00}(\Gamma_c)]'; \ f \text{ and } \phi \text{ are compatible}\}.$$

Then:

(i) M is dense in $[H^{1/2}_{00}(\Gamma_c)]'$.

(ii) In the case of compatible data, the Cauchy problem (1) admits a unique solution in $H^1(D)$.

4. Nash game formulation of the Cauchy problem

We present an algorithm based on a game theory. This algorithm was introduced in [13] for solving the Cauchy problem for an elliptic linear operator. We extend it to solve our nonlinear Cauchy problem (1), where the function k is given by (2). The Cauchy problem is formulated as a two-player. The first player is given the known Dirichlet data f and uses the Neumann condition prescribed over the inaccessible Γ_i part of the boundary ∂D as strategy variable. The second player is given the known Neumann data ϕ, and plays with the Dirichlet condition prescribed over the inaccessible boundary. Following the work in [13], for all $(\eta, \tau) \in (H^{\frac{1}{2}}_{00}(\Gamma_i))' \times H^{\frac{1}{2}}(\Gamma_i)$, we define

$$\begin{cases} J_1(\eta,\tau) = \frac{1}{2}\|(k(|\nabla u_1(\eta)|^2)\nabla u_1(\eta)\cdot n - \phi\|^2_{(H^{\frac{1}{2}}_{00}(\Gamma_c))'} + \frac{1}{2}\|u_1(\eta) - u_2(\tau)\|^2_{H^{\frac{1}{2}}(\Gamma_i)} \\ J_2(\eta,\tau) = \frac{1}{2}\|u_2(\tau) - f\|^2_{H^{\frac{1}{2}}(\Gamma_c)} + \frac{1}{2}\|u_1(\eta) - u_2(\tau)\|^2_{H^{\frac{1}{2}}(\Gamma_i)} \end{cases}$$

where $u_1(\eta)$ and $u_2(\tau)$ are the solutions to:

$$\begin{cases} \nabla\cdot[k(|\nabla u_1|^2)\nabla u_1] = 0, & \text{in } D, \\ u_1 = f, & \text{on } \Gamma_c, \\ k(|\nabla u_1|^2)\nabla u_1 \cdot n = \eta, & \text{on } \Gamma_i, \end{cases} \tag{SP_1}$$

and

$$\begin{cases} \nabla\cdot[k(|\nabla u_2|^2)\nabla u_2] = 0, & \text{in } D, \\ u_2 = \tau, & \text{on } \Gamma_i, \\ k(|\nabla u_1|^2)\nabla u_2 \cdot n = \phi, & \text{on } \Gamma_c. \end{cases} \tag{SP_2}$$

The two players solve in parallel the associated boundary value problems (SP_1) and (SP_2). We seek to find a couple $(\eta_c; \tau_c)$, called “Pareto optimal Nash equilibrium" in the game theory vocabulary.

Definition 4.1 *A pair $(\eta_N, \tau_N) \in (H^{\frac{1}{2}}_{00}(\Gamma_i))' \times H^{\frac{1}{2}}(\Gamma_i)$ is called:*

– Nash equilibrium for the two-players game involving the costs functionals J_1 and J_2 if:

$$J_1(\eta_N, \tau_N) \leq J_1(\eta, \tau_N), \ \forall\, \eta \in (H^{\frac{1}{2}}_{00}(\Gamma_i))',$$

$$J_2(\eta_N, \tau_N) \leq J_1(\eta_N, \tau), \ \forall\, \tau \in H^{\frac{1}{2}}(\Gamma_i).$$

– Pareto optimal Nash equilibrium for the two-players game involving the costs J_1 and J_2 if there does not exist another Nash equilibrium $(\eta_n, \tau_n) \in (H^{\frac{1}{2}}_{00}(\Gamma_i))' \times H^{\frac{1}{2}}(\Gamma_i)$ such that:

$$J_1(\eta_n, \tau_n) \leq J_1(\eta_N, \tau_N) \text{ and } J_2(\eta_n, \tau_n) \leq J_1(\eta_N, \tau_N).$$

Proposition 2 *When the Cauchy problem (1) has a solution u, the pair*

$$(\eta_c, \tau_c) = (k(|\nabla u|^2)\nabla u \cdot n|_{\Gamma_i}, u|_{\Gamma_i})$$

is unique Pareto optimal Nash equilibrium for the two-players game involving the costs functionals J_1 and J_2.

Remark *(i) For the case $k(\cdot) \equiv 1$ we have a Cauchy problem for the Laplace operator. Nash game formulation of this problem was studied in [13]. The costs functionals are elliptic and convex. The authors proved that there always exists a unique Nash equilibrium, which turns out to be the reconstructed data when the Cauchy problem has a solution. They also proved that the completion algorithm is stable with respect to noise.*

(ii) Linear model gives a smooth solution. Hence it is unable to restore edges and we must take care of edges in image since they are crucial for object recognition and image processing problems. The use of a the nonlinear model defined by (1) and (2) allows us to inpaint edges while keeping the edges.

4.1. Numerical procedure

To compute the Nash equilibrium, we use the classical alternating minimization algorithm with relaxation, by means of the following iterative process:

– Set $k = 0$ and choose an initial guess $S^{(0)} = (\eta^{(0)}, \tau^{(0)})$.

– Step 1: Compute $\overline{\eta}^{(k)}$ solution of

$$\min_{\eta} \; J_1(\eta, \tau^{(k)}).$$

– Step 2: Compute $\overline{\tau}^{(k)}$ solution of

$$\min_{\tau} J_2(\eta^{(k)}, \tau).$$

– Step 3: Set $S^{(k+1)} = (\eta^{(k+1)}, \tau^{(k+1)}) = t(\eta^{(k)}, \tau^{(k)}) + (1-t)(\overline{\eta}^{(k)}; \overline{\tau}^{(k)})$, $0 < t < 1$.

– If $||S^{(k+1)} - S^{(k)}|| \leq \epsilon$, stop. Otherwise $k = k + 1$, go to Step 1.

The gradients may be efficiently computed by means of an adjoint state method and we have the following two partial derivatives:

$$\frac{\partial J_1}{\partial \eta}(\eta, \tau)h = -\int_{\Gamma_i} \lambda_1 h \, ds, \; \forall h \in (H^{\frac{1}{2}}_{00}(\Gamma_i))',$$

where $\lambda_1 \in W = \{u \in H^1(D); \; u|_{\Gamma_c} = 0\}$ solves the adjoint problem

$$\int_D B(\nabla u_1)\nabla\sigma \cdot \nabla\lambda_1 \, dx = \int_{\Gamma_c} (k(|\nabla u_1|^2)\nabla u_1 \cdot n - \phi)(B(\nabla u_1)\nabla\sigma \cdot n) \, ds$$
$$+ \int_{\Gamma_i} (u_1 - u_2)\sigma \, ds, \; \forall \sigma \in W.$$

and

$$\frac{\partial J_2}{\partial \tau}(\eta,\tau)\xi = \int_{\Gamma_i} (B(\nabla u_2)\nabla\lambda_2 \cdot n + u_2 - u_1)\xi\, ds,\ \forall \xi \in H^{\frac{1}{2}}(\Gamma_i),$$

where λ_2 solves the adjoint problem

$$\begin{cases} \nabla \cdot (B(\nabla u_2)\nabla\lambda_2) = 0 & \text{in } \Omega \\ \lambda_2 = 0 & \text{on } \Gamma_i \\ (B(\nabla u_2)\nabla\lambda_2)\cdot n = f - u_2 & \text{on } \Gamma_c \end{cases}$$

with

$$B(\omega) = k(|\omega|^2)I_2 + 2k'(|\omega|^2)\begin{pmatrix} \omega_1^2 & \omega_1\omega_2 \\ \omega_1\omega_2 & \omega_2^2 \end{pmatrix},\ \ \omega = (\omega_1,\omega_2).$$

5. Numerical results

All the numerical results were obtained by the finite-element method which were implemented in the Freefem++ Software environment. We illustrate the numerical results obtained using the algorithm described in the previous section. To evaluate the effectiveness of the proposed method, we tested it for some examples for exact and noisy data. In all the numerical tests, we have chosen images where the missing region D is such that $\partial D \cap \partial\Omega \neq \emptyset$, see Fig.1 (b).

Linear case: The first test case, a rather simple, has been shown in Fig. 2. It is a direct application of the model proposed in [13] for Laplace Cauchy problem and were performed on a smooth image for noisy Cauchy data. We compare this result to those given by the total variation inpainting, which uses the homogeneous Neumann condition.

Nonlinear case: The second result concerns the nonlinear model, where we have chosen images containing edges and jumps. We have tested different values of α in our algorithm. We illustrate in Fig. 3 the results obtained using our algorithm (for $\alpha = 1$ and 10^{-6})). To see the efficiency of the proposed method from data completion point of view, we illustrate in Fig. 4 the numerical Dirichlet and Neumann solutions compare it with the original image for different values of α. The numerical Dirichlet solution remains good (see the top plot of Fig. 4) for $\alpha = 10^{-6}$. On the other hand, concerning the numerical Neumann solution (see the right-hand plot of Fig. 4), it can be seen that we have two picks which correspond to edges in the image.

6. Conclusion

In this paper, we have investigated a Cauchy problem for a nonlinear elliptic equation in image inpainting. This Cauchy problem was introduced to treat the case when a Dirichlet boundary condition is not known on a part on the boundary ∂D and was formulated as a Nash game, where Dirichlet data f uses the Neumann condition η prescribed over the inaccessible Γ_i part of the boundary ∂D as strategy variable to play against Neumann data ϕ which uses in turn the Dirichlet condition τ prescribed over the inaccessible part of the boundary. Numerical experiments on different images were performed and showed the efficiency of the proposed method. This work can be extended in many different ways. For instance, one might consider using our approach for other inpainting models

such as the one based on Ginzberg-Landau equation. One can also theoretically study the convergence of the numerical method used for solving the Nash-game formulation.

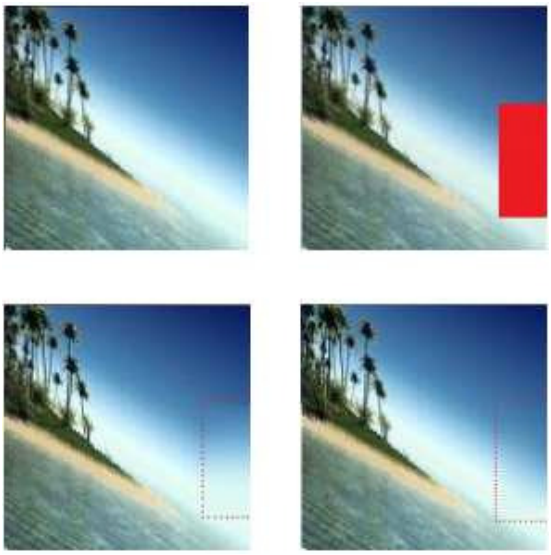

Figure 2. *Top row: original, damaged. Bottom row: TV* $(MSE = 7.73\,10^{-3})$*, our method* $(MSE = 4.63\,10^{-4})$.

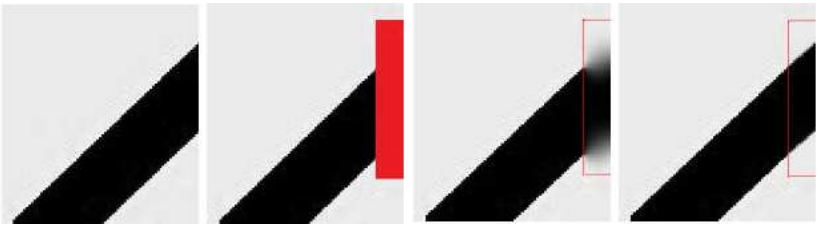

Figure 3. *From left to right: original, damaged and Nash game for* $\alpha = 1$ *and* $\alpha = 10^{-8}$.

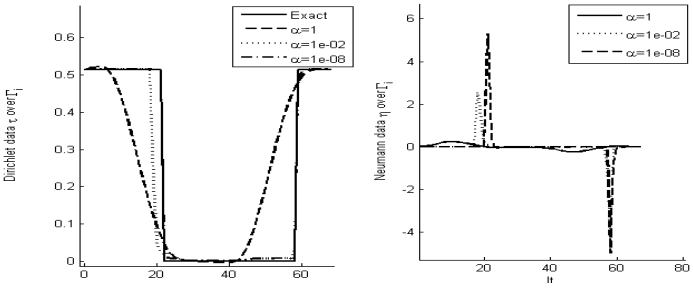

Figure 4. *Reconstructed Dirichlet* (τ_N*, left) and Neumann* (η_N*, right) data over* Γ_i.

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