Second-order topological expansion with respect to the insertion of small coated inclusion

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Abstract. In this paper, we consider the inverse problem of recovering coated inclusions from boundary measurements. To solve the inverse problem numerically, we propose a Kohn-Vogelius type functional and we perform its first as well as its second-order topological gradient.
The first-order topological gradient usually involves the state and the adjoint solutions and their gradients estimated at the point where the topological perturbation is performed. In the case of second-order topological gradient, non-local terms which have a higher computational cost appear. In this work, we aime at determining the relevance of these non-local terms from the numerical point of view.

Keywords: Inverse problem,Topological optimization, Second-order asymptotic analysis, Coated inclusion

1 Introduction

In this work, we consider the inverse problem of locating coated inclusions from boundary measurements. The main motivations of this problem are non-destructive testing of a conductor bodies in order to reconstruct electric conductivity inside a hidden objects, geophysical and medical applications.

A widely used solution approach to solve the inverse problem from given measurement data is the topological gradient method. The main idea in this approach is to perform a sensitivity analysis of a given shape functional with respect to the insertion of a small hole inside the domain. More precisely, we consider a domain $\Omega \subset \mathbb{R}^2$ and a cost functional $j(\Omega) = J(u_\Omega)$, were u_Ω is the state variable, i.e. a solution of a given partial differential equation in Ω. For $\varepsilon > 0$, let $\Omega_\varepsilon = \Omega \setminus \overline{(x_0 + \varepsilon B)}$ be the domain obtained by removing a small part $\overline{(x_0 + \varepsilon B)}$ from Ω, at a location $x_0 \in \Omega$, and B is a fixed bounded domain containing the origin (e.g. the unit ball in $\mathbb{R}^2$). Then, the first order asymptotic expansion of the functional j with respect to ε, takes the form

$$j(\Omega_\varepsilon) - j(\Omega) = f(\varepsilon) D_T j(x_0) + o(f(\varepsilon)), \tag{1}$$

where $f(\varepsilon)$ is a positive function going to zero with ε and $D_T j$ is the so called topological gradient. Therefore, to minimize the cost functional j, one has to create small holes at the locations x where $D_T j(x)$ is the most negative. For detailed description of this approach we refer the reader to [1,2,3] and the references therein.

If we assume that the shape functional j admits the following topological asymptotic expansion

$$j(\Omega_\varepsilon) - j(\Omega) = f_1(\varepsilon) D_T j(x_0) + f_2(\varepsilon) D_T^2 j(x_0) + o(f_2(\varepsilon)), \qquad (2)$$

where $f_1(\varepsilon)$ and $f_2(\varepsilon)$ are positive and smooth functions that decreases monotonically such that $f_1(\varepsilon) \to 0$ when $\varepsilon \to 0_+$ and

$$\lim_{\varepsilon \to 0} \frac{f_2(\varepsilon)}{f_1(\varepsilon)} = 0, \qquad \lim_{\varepsilon \to 0} \frac{o(f_2(\varepsilon))}{f_2(\varepsilon)} = 0,$$

then, $D_T j$ and $D_T^2 j$ are the first and second order topological derivatives of j, respectively.

The topological asymptotic expansion has been derived for various operators and has been applied for many applications[4,5]. The numerical algorithms for the reconstruction procedure are in general based on the first-order topological derivative which is only valid for infinitesimal geometry perturbation. When dealing with perturbation of finite size, the second-order topological derivate may be decisive to improve the numerical algorithms and avoid being trapped in a local minima.

In this work, we derive the second-order topological asymptotic expansion of a Kohn-Vogelius type functional with respect to the insertion of small circular coated inclusion.

The case of the insertion of simple inclusion was considered by Moskow and Vogelius[6,7]. It consists in perturbing the domain by inserting a small inclusion with a different material property from the background medium.

In this work, the discontinuous conductivity in the inclusion itself prevents from such methods and results based on a local perturbation of the material properties. Therefore, we extend the sensitivity analysis to the problem under consideration and we perform the scond-order asymptotic expansion of the proposed shape functionals using non-standard perturbation techniques.

2 The model problem

Let Ω be a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$ and let ω be an open subset of Ω composed of two different subdomains Ω_1 and Ω_2 where the subset Ω_2 is surrounded by the subset Ω_1. We denote by $\Gamma_2 := \partial\Omega_2$ and $\Gamma_1 \cup \Gamma_2 := \partial\Omega_1$ as depicted in Figure 1 and we set $\Omega_0 := \Omega \backslash \overline{\Omega_1 \cup \Omega_2}$. Throughout the article, we consider piecewise constant conductivity

$$\sigma = \sigma_0 \mathbb{1}_{\Omega_0} + \sigma_1 \mathbb{1}_{\Omega_1} + \sigma_2 \mathbb{1}_{\Omega_2},$$

where $\sigma_0, \sigma_1, \sigma_2 \in \mathbb{R}_+^*$ and $\mathbb{1}_E$ denotes the indicator function of the set E. We assume further that there exists two constants c_0, c_1 such that

$$0 < c_0 \leq \sigma_0, \sigma_1, \sigma_2 \leq c_1$$

For a given current density $g \in H^{-1/2}(\partial\Omega)$ verifying the compatibility condition

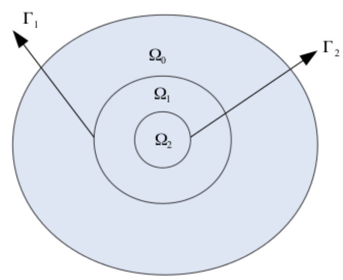

Fig. 1. The domain $\Omega = \Omega_0 \cup \Omega_1 \cup \Gamma_1 \cup \Gamma_2 \cup \Omega_2$.

$$\langle g, 1\rangle_{H^{-1/2}(\partial\Omega), H^{1/2}(\partial\Omega)} = 0,$$

the potential u_ω generated by the Neumann data g (in the presence of the coated inclusion ω) satisfy the following problem:

$$\begin{cases} \operatorname{div}(\sigma\nabla u_\omega) = 0 \quad \text{in } \Omega_0 \cup \Omega_1 \cup \Omega_2, \\ [\![u_\omega]\!] = 0 \quad \text{on } \Gamma_i, i = 1, 2, \\ [\![\sigma\partial_\nu u_\omega]\!] = 0 \quad \text{on } \Gamma_i, i = 1, 2, \\ \sigma_0 \partial_\nu u_\omega = g \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where, ν is the outward unit normal vector to $\partial\Omega$ and

$$[\![u]\!] := u^+ - u^-, \text{ with } u^+ = \gamma_0\big|_{\Omega_{i-1}}(u), \quad u^- = \gamma_0\big|_{\Omega_i}(u),$$

$$[\![\partial_\nu u]\!] := \partial_\nu u^+ - \partial_\nu u^-, \text{ with } \partial_\nu u^+ = \gamma_1\big|_{\Omega_{i-1}}(u), \quad \partial_\nu u^- = \gamma_1\big|_{\Omega_i}(u).$$

Here $\gamma_0\big|_{\Omega_i} : H^1(\Omega_i) \to H^{1/2}(\Gamma_i)$, $\gamma_1\big|_{\Omega_i} : H(\Omega_i) \to H^{-1/2}(\Gamma_i)$ are trace operators and $H(\Omega_i)$ is a subspace of $H^1(\Omega_i)$ defined by

$$H(\Omega_i) := \left\{u \in H^1(\Omega_i) : \Delta u \in L^2(\Omega_i)\right\}.$$

From the Riesz representation theorem, we can prove that problem (3) has a unique solution $u \in H^1(\Omega)$. The inverse problem we consider here is is the following.

Given the Neumann data g and the potential $u|_{\partial\Omega} := f$ measured on $\partial\Omega$, find the location of the coated inclusion $\omega \subset \Omega$. (4)

A typical approach to solve the inverse problem in practice is to consider the so-called *Kohn-Vogelius functional*. The corresponding minimization problem is:

$$\begin{cases} \text{minimize } j(\omega) := J(u_\omega^N, u_\omega^D) := \dfrac{1}{2}\displaystyle\int_\Omega \sigma|\nabla(u_\omega^N - u_\omega^D)|^2\, dx \\ \text{subject to } \omega \in \mathcal{O}, \end{cases} \tag{5}$$

where

$$\mathcal{O} = \{\omega \subset \Omega, \omega \text{ open Lipschitz and } \operatorname{dist}(\omega, \partial\Omega) \geq c_0, \text{ for some } c_0 > 0\}.$$

Here u_ω^N is the solution of the Neumann problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u_\omega^N) = 0 & \text{in } \Omega_0 \cup \Omega_1 \cup \Omega_2, \\ [\![u_\omega^N]\!] = 0 & \text{on } \Gamma_i, \\ [\![\sigma\partial_\nu u_\omega^N]\!] = 0 & \text{on } \Gamma_i, \\ \sigma_0\partial_\nu u_\omega^N = g & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} u_\omega^N\, ds = 0, \end{cases} \tag{6}$$

and u_ω^D is the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma\nabla u_\omega^D) = 0 & \text{in } \Omega_0 \cup \Omega_1 \cup \Omega_2, \\ [\![u_\omega^D]\!] = 0 & \text{on } \Gamma_i, i = 1, 2, \\ [\![\sigma\partial_\nu u_\omega^D]\!] = 0 & \text{on } \Gamma_i, i = 1, 2 \\ u_\omega^D = f & \text{on } \partial\Omega. \end{cases} \tag{7}$$

The weak formulations of the problems (6) and (7) read respectively

$$a^N(u_\omega^N, v) = l^N(v), \quad v \in \mathcal{V}^N, \tag{8}$$

$$a^D(u_\omega^D, v) = l^D(v), \quad \mathcal{V}^D, \tag{9}$$

where

$$a^N(u_\omega^N, v) = \int_\Omega \sigma\nabla u_\omega^N \cdot \nabla v\, dx, \quad l^N(v) \int_{\partial\Omega} gv\, ds,$$

$$a^D(u_\omega^D, v) = \int_\Omega \sigma\nabla u_\omega^N \cdot \nabla v\, dx, \quad l^D(v) = \int_{\partial\Omega} (u_\omega^D - f)\partial_\nu v\, ds,$$

$$\mathcal{V}^N(\Omega) = \left\{ v \in H^1(\Omega) : \int_\Omega v\, dx = 0 \right\},$$

and

$$\mathcal{V}^D(\Omega) = \left\{ v \in H_0^1(\Omega) : \operatorname{div}(\sigma\nabla v) \in L^2(\Omega) \right\}.$$

We denote by u_0^N, u_0^D, the background solutions:

$$\Delta u_0^N = 0 \text{ in } \Omega, \quad \sigma_0\partial_\nu u_0^N = g \text{ in } \partial\Omega, \quad \int_\Omega u_0^N\, dx = 0, \tag{10}$$

$$\Delta u_0^D = 0 \text{ in } \Omega, \quad u_0^D = f \text{ in } \partial\Omega. \tag{11}$$

3 Topological derivatives

Now, we assume that the domain $\Omega_2 := x_0 + \alpha\varepsilon B$ and Ω_1 is such that $\Omega_1 \cup \Omega_2 = x_0 + \varepsilon B$, where B is the unit ball in $\mathbb{R}^2$, $\varepsilon > 0$ and $0 < \alpha < 1$. We rewrite Ω_1^ε and Ω_2^ε instead of Ω_1 and Ω_2. This allows to perform an asymptotic expansion of the shape functional $J_p(\omega_\varepsilon)$ where $\omega_\varepsilon := \Omega_2^\varepsilon \cup \Omega_1^\varepsilon$. We also introduce $\Gamma_2^\varepsilon := \partial\Omega_2^\varepsilon$ and Γ_1^ε the outer boundary of Ω_1^ε. In the perturbed domain, the state u_ε^N and u_ε^D are respectively solutions of the following problems:

$$\operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon^N) = 0 \text{ in } \Omega, \quad \sigma_0 \partial_\nu u_\varepsilon^N = g \text{ in } \partial\Omega, \quad \int_\Omega u_\varepsilon^N \, ds = 0, \tag{12}$$

$$\operatorname{div}(\sigma_\varepsilon \nabla u_\varepsilon^D) = 0 \text{ in } \Omega, \quad u_\varepsilon^D = f \text{ in } \partial\Omega, \tag{13}$$

where

$$\sigma_\varepsilon = \sigma_2 \mathbb{1}_{\Omega_2^\epsilon} + \sigma_1 \mathbb{1}_{\Omega_1^\epsilon} + \sigma_0 \mathbb{1}_{\Omega \setminus \overline{\Omega_1^\epsilon \cup \Omega_2^\epsilon}},$$

The weak formulations to the problems (12) and (13) are given by:

$$a_\varepsilon^N(u_\varepsilon^N, v) = l_\varepsilon^N(v), \quad v \in \mathcal{V}^N(\Omega), \tag{14}$$

$$a_\varepsilon^D(u_\varepsilon^D, v) = l_\varepsilon^D(v), \quad v \in \mathcal{V}_\varepsilon^D(\Omega), \tag{15}$$

where

$$a_\varepsilon^N(u_\varepsilon^N, v) = \int_\Omega \sigma_\varepsilon \nabla u_\varepsilon^N \cdot \nabla v \, dx, \quad l_\varepsilon^N(v) = \int_{\partial\Omega} g v \, ds, \tag{16}$$

$$a_\varepsilon^D(u_\varepsilon^D, v) = \int_\Omega \sigma_\varepsilon \nabla u_\varepsilon^D \cdot \nabla v \, dx, \quad l_\varepsilon^D(v) = \int_{\partial\Omega} (u_\varepsilon^D - f) \sigma_0 \partial_\nu v \, ds, \tag{17}$$

and

$$\mathcal{V}_\varepsilon^D(\Omega) = \left\{ v \in H_0^1(\Omega) : \operatorname{div}(\sigma_\varepsilon \nabla v) \in L^2(\Omega) \right\}. \tag{18}$$

The adjoint states v_ε^N and v_ε^D are characterized by

$$a_\varepsilon^N(\phi, v_\varepsilon^N) = -\partial_{u_\varepsilon^N} J(u_\varepsilon^N, u_\varepsilon^D)\phi, \quad \phi \in \mathcal{V}^N(\Omega),$$

$$a_\varepsilon^D(\phi, v_\varepsilon^D) = -\partial_{u_\varepsilon^D} J(u_\varepsilon^N, u_\varepsilon^D)\phi, \quad \phi \in \mathcal{V}_\varepsilon^D(\Omega).$$

Or equivalently

$$\operatorname{div}(\sigma_\varepsilon \nabla v_\varepsilon^N) = 0, \text{ in } \Omega, \quad \partial_\nu v_\varepsilon^N = -g \text{ on } \partial\Omega,$$

$$\operatorname{div}(\sigma_\varepsilon \nabla v_\varepsilon^D) = 0, \text{ in } \Omega, \quad v_\varepsilon^D = 0 \text{ on } \partial\Omega.$$

This imply that $v_\varepsilon^N = -u_\varepsilon^N$ and $v_\varepsilon^D = 0$. We introduce the following proposition which describes a method for the computation of the second-order variation of a given shape functional.

Proposition 1. *Let $\mathcal{V}$ be a Hilbert space. For $\varepsilon \in [0,\xi), \xi > 0$, consider a function $u_\varepsilon \in \mathcal{V}$ solving a variational problem of the form*

$$a_\varepsilon(u_\varepsilon, v) = l_\varepsilon(v) \quad \forall v \in \mathcal{V}, \tag{19}$$

where a_ε and l_ε are a bilinear form and a linear form on $\mathcal{V}$, respectively. For all $\varepsilon \in [0,\xi)$, consider a functional $j(\varepsilon) = J_\varepsilon(u_\varepsilon)$ where $J_\varepsilon : \mathcal{V} \longrightarrow \mathbb{R}$ is Frêchet differentiable at the point u_0. Suppose that the following hypotheses hold.

$\mathbf{H_1}$ *There exist numbers $\delta a, \delta^2 a$, $\delta l, \delta^2 l$ and functions $f_1(\varepsilon), f_2(\varepsilon) \geq 0$ such that*

$$(a_\varepsilon - a_0)(u_0, v_\varepsilon) = f_1(\varepsilon)\delta a + f_2(\varepsilon)\delta^2 a + o(f_2(\varepsilon)), \tag{20}$$

$$(l_\varepsilon - l_0)(v_\varepsilon) = f_1(\varepsilon)\delta l + f_2(\varepsilon)\delta^2 l + o(f_2(\varepsilon)), \tag{21}$$

$$\lim_{\varepsilon\to 0} f_1(\varepsilon) = 0, \quad \lim_{\varepsilon\to 0}\frac{f_2(\varepsilon)}{f_1(\varepsilon)} = 0, \quad \lim_{\varepsilon\to 0}\frac{o(f_2(\varepsilon))}{f_2(\varepsilon)} = 0, \tag{22}$$

where $v_\varepsilon \in \mathcal{V}$ is an adjoint state satisfying

$$a_\varepsilon(\varphi, v_\varepsilon) = -DJ_\varepsilon(u_0)\varphi, \quad \forall \varphi \in \mathcal{V}. \tag{23}$$

$\mathbf{H_2}$ *There exist numbers $\delta J_i, \delta^2 J_i, \quad i = 1,2$, such that*

$$J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) - DJ_\varepsilon(u_0)(u_\varepsilon - u_0) = f_1(\varepsilon)\delta J_1 + f_2(\varepsilon)\delta^2 J_1 + o(f_2(\varepsilon)), \tag{24}$$

$$J_\varepsilon(u_0) - J_0(u_0) = f_1(\varepsilon)\delta J_2 + f_2(\varepsilon)\delta^2 J_2 + o(f_2(\varepsilon)). \tag{25}$$

Then the second-order variation of the cost function with respect to ε is given by

$$\begin{aligned} j(\varepsilon) - j(0) =& f_1(\varepsilon)(\delta a - \delta l + \delta J_1 + \delta J_2) \\ &+ f_2(\varepsilon)(\delta^2 a - \delta^2 l + \delta^2 J_1 + \delta^2 J_2) + o(f_2(\varepsilon)). \end{aligned} \tag{26}$$

Theorem 1. *The first order topological asymptotic expansion of the functional J with respect to the insertion of small coated inclusion ω_ε is given by*

$$J_\varepsilon(u_\varepsilon^N, u_\varepsilon^D) - J_0(u_0^N, u_0^D) = \varepsilon^2 G(x_0) + o(\varepsilon^2),$$

where

$$G(x_0) = (\delta a - \delta l + \delta J_1 + \delta J_2). \tag{27}$$

such that

$$\delta a = \pi |\nabla u_0^N(x_0)|^2 \left((\sigma_0 - \sigma_1)(1-\alpha^2)(\beta+1) + \alpha^2(\sigma_0 - \sigma_2)(\beta+\gamma+1)\right) \tag{28}$$

and

$$\delta l = \pi u_0^N(x_0)\left((f_0 - f_1)(1-\alpha^2) + \alpha^2(f_0 - f_2)\right) = 0 \text{ due to } f_0 = f_1 = f_2 \tag{29}$$

and

$$\delta J_1 = \frac{\pi}{2}|\nabla(u_0^N(x_0) - u_0^D(x_0))|^2 \left(\alpha^2\sigma_2(\beta+\gamma)^2 + \sigma_1(1-\alpha^2)(\beta^2 + (\alpha\gamma)^2) + \sigma_0(\beta+\gamma\alpha^2)^2\right) \tag{30}$$

and finally

$$\delta J_2 = \frac{\pi}{2}\varepsilon^2\left((1-\alpha^2)(\sigma_1 - \sigma_0) + \alpha^2(\sigma_2 - \sigma_0)\right)|\nabla(u_0^N(x_0) - u_0^D(x_0))|^2 \tag{31}$$

Theorem 2. *The second topological asymptotic expansion of the functional* J *with respect to the insertion of small coated inclusion* ω_ε *is given by*

$$J_\varepsilon(u_\varepsilon^N, u_\varepsilon^D) - J_0(u_0^N, u_0^D) - \varepsilon^2 G(x_0) = \varepsilon^4 G^2(x_0, \varepsilon) + o(\varepsilon^4). \tag{32}$$

where

$$G^2(x_0) = \delta a_1 - \delta l_1 + \delta J_{11} + \delta J_{22} \tag{33}$$

with

$$\begin{aligned}
\delta a_1 = {}& \frac{\pi}{4}\|\nabla^2 u_0^N(x_0)\|_F^2 \left((\sigma_0-\sigma_1)(1-\alpha^4)(\beta_1+1) + (\sigma_0-\sigma_2)\left(\beta_1+\gamma_1+1\right)\alpha^4\right)\\
&- \frac{\pi}{8}\frac{|\Delta u_0^N(x_0)|^2}{2}\left((\sigma_0-\sigma_1)(1-\alpha^4)\left(\beta_1 - \frac{2(\sigma_1-\sigma_2)}{\sigma_1(1+\alpha^2)}\right) + (\sigma_0-\sigma_2)\left(\beta_1+\gamma_1\right)\alpha^6\right)\\
&- \frac{\pi}{8}\left((1-\alpha^4)(\sigma_0-\sigma_1)\beta_2 + \alpha^4(\sigma_0-\sigma_2)(\beta_2+\gamma_2)\right)\nabla u_0^N(x_0)\cdot(\Delta(\nabla u_0^N(x_0)+2\nabla(\Delta u_0^N(x_0))\\
&- \frac{\pi}{8}(\sigma_0-\sigma_1)\gamma\alpha^2(1-\alpha^2)\nabla(\Delta u_0^N(x_0))\cdot\nabla u_0^N(x_0)\\
&+ \frac{\pi}{8}\Delta(\nabla u_0^N(x_0))\cdot\nabla u_0^N(x_0)\left((\beta+\gamma)\alpha^4(\sigma_0-\sigma_2) + \left(\beta(1-\alpha^4)+\gamma\alpha^2(1-\alpha^2)\right)(\sigma_0-\sigma_1)\right)\\
&+ \pi(\beta+\alpha^2\gamma)|\nabla u_0^N(x_0)|^2\left((\sigma_0-\sigma_1)(1-\alpha^2)+(\sigma_0-\sigma_2)\alpha^2\right)
\end{aligned} \tag{34}$$

with β, γ, β_1 *and* γ_1 *are defined respectively in* (35), (36), (37), (38).

$$\beta := \frac{(\sigma_1+\sigma_2)(\sigma_0-\sigma_1) - \alpha^2(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}{(\sigma_1+\sigma_2)(\sigma_1+\sigma_0) + \alpha^2(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}, \tag{35}$$

$$\gamma := \frac{2\sigma_0(\sigma_1-\sigma_2)}{(\sigma_1+\sigma_2)(\sigma_1+\sigma_0) + \alpha^2(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}. \tag{36}$$

$$\beta_1 := \frac{(\sigma_1+\sigma_2)(\sigma_0-\sigma_1) - \alpha^4(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}{(\sigma_1+\sigma_2)(\sigma_1+\sigma_0) + \alpha^4(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}, \tag{37}$$

$$\gamma_1 := \frac{2\sigma_0(\sigma_1-\sigma_2)}{(\sigma_1+\sigma_2)(\sigma_1+\sigma_0) + \alpha^4(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}. \tag{38}$$

For the expansion of the cost function we have

$$J_\varepsilon(u_\varepsilon^N, u_\varepsilon^D) - J_\varepsilon(u_0^N, u_0^D) - DJ_\varepsilon(u_0^N, u_0^D)(u_\varepsilon^N - u_0^N, u_\varepsilon^D - u_0^D) = \varepsilon^4 \delta J_{11} + \varepsilon^2 \delta J_1 + o(\varepsilon^4) \tag{39}$$

with

$$\begin{aligned}
\delta J_{11} = {}& \frac{\pi}{8}\left(\|\nabla^2(u_0^N(x_0)-u_0^D(x_0))\|_F^2 \times \mathbb{Q}_1 + |\Delta(u_0^N(x_0)-u_0^D(x_0))|^2 \times \mathbb{Q}_2\right)\\
&+ \frac{\pi}{8}(\Delta(\nabla(u_0^N(x_0)-u_0^D(x_0)) + 2\nabla(\Delta(u_0^N(x_0)-u_0^D(x_0)))\cdot\nabla(u_0^N(x_0)-u_0^D(x_0))\times\mathbb{P}\\
&+ \frac{\pi}{2}(\beta+\alpha^2\gamma)^2\sigma_0|\nabla(u_0^N(x_0)-u_0^D(x_0))|^2\\
&+ \pi(\beta+\alpha^2\gamma)\left(\sigma_2\alpha^2(\beta+\gamma)+\sigma_1(1-\alpha^2)\beta\right)\left(|\nabla u_0^N(x_0)|^2 - |\nabla u_0^D(x_0))|^2\right)
\end{aligned} \tag{40}$$

with δJ_1 defined in (30) $\mathbb{Q}_1$, $\mathbb{Q}_2$ *and* $\mathbb{P}$ *defined in* (44), (45) *and* (43) *respectively and* β_2, γ_2 *defined in* (41) *and* (42) *respectively.*

$$\beta_2 := \frac{(\sigma_1+\sigma_2)(\sigma_0-\sigma_1) - \alpha^4(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}{(\sigma_1+\sigma_2)(\sigma_1+\sigma_0) + \alpha^2(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}, \tag{41}$$

$$\gamma_2 := \frac{(\sigma_0-\sigma_1)(\sigma_1-\sigma_2) + \alpha^2(\sigma_0+\sigma_1)(\sigma_1-\sigma_2)}{(\sigma_1+\sigma_2)(\sigma_1+\sigma_0) + \alpha^2(\sigma_1-\sigma_2)(\sigma_0-\sigma_1)}. \tag{42}$$

$$\mathbb{P} = \left(\sigma_0(\beta_2+\alpha^2\gamma_2)(\beta+\alpha^2\gamma) + \sigma_1\left[(1-\alpha^2)(\beta\beta_2+\alpha^2\gamma\gamma_2)\right] + \sigma_2(\beta_2+\gamma_2)(\beta+\gamma)\alpha^2\right). \tag{43}$$

$$\mathbb{Q}_1 = \sigma_0(\beta_1+\alpha^4\gamma_1)^2 + \sigma_1(1-\alpha^4)(\beta_1^2+4\alpha^4\gamma_1^2) + \sigma_2\left(\alpha^4(\beta_1+\gamma_1)^2\right) \tag{44}$$

$$\mathbb{Q}_2 = \frac{1}{2}\sigma_0(\beta_1+\alpha^4\gamma_1)^2 + \sigma_2\alpha^4(\beta_1+\gamma_1)^2\left(\frac{\alpha^2(\alpha^2-2)}{2}\right)$$
$$+\sigma_1\left((1-\alpha^4)(\frac{3\alpha^4\gamma_1^2-\beta_1^2}{2}) + 4(1-\alpha^2)\alpha^2(\frac{\sigma_0-\sigma_1}{\sigma_1})\gamma_1 + \alpha^4\log(\alpha)(8\beta_1\gamma_1 - (\frac{\sigma_1-\sigma_2}{\sigma_1})^2)\right) \tag{45}$$

In addition we have

$$J_0(u_\varepsilon^N, u_\varepsilon^D) - J_0(u_0^N, u_0^D) = \varepsilon^4\delta J_{22} + \varepsilon^2\delta J_2 + o(\varepsilon^4) \tag{46}$$

with

$$\delta J_{22} = \frac{\pi}{8}\left((\sigma_1-\sigma_0)(1-\alpha)^4 + (\sigma_2-\sigma_0)\alpha^4\right)\|\nabla^2(u_0^N(x_0) - u_0^D(x_0))\|_F^2 \tag{47}$$

and δJ_2 defined in (31).

4 Numerical results

In this section, we present some numerical results using the first and the second order toplogical derivative expansion expressed in (27) and (33). We aim to obtain a similtaneous suitable approximation of the size and the position of the coated inclusion.
The direct and the adjoint problems are solved using the finite element method. In order to find the size and the position of the coated inclusion, we follow the idea in [8] and we associate it to the algorithm used in[9].

Remark 1. The toplogical asymptotic expansion takes the following quadratic form with respect to ζ the size of the inclusion:

$$\mathcal{J}(\Omega_\varepsilon) = \mathcal{J}(\Omega) + \zeta g(x) + \frac{1}{2}H(x)\zeta\cdot\zeta + \mathcal{E}(\varepsilon). \tag{48}$$

where $\zeta = \varepsilon^2$ and $\mathcal{E}(\varepsilon) = o(\varepsilon^4)$.
To find an optimal ζ we differenciate(48) with respect to ζ to obtain the first order optimality condition

$$H(x)\zeta = -g(x) \tag{49}$$

The quantity ζ solving becomes a function of the locations $x, \zeta = \zeta(x)$.
Therefore, the optimal locations x can be obtained from a combinatorial search over the domain Ω as solutions to the following minimization problem:

$$\overset{\star}{x} = \underset{x \in \Omega}{\operatorname{argmin}} \left\{ \zeta(x) = -\frac{g(x)}{H(x)} \right\} \tag{50}$$

Finally the optimal size of the inclusion is $\overset{\star}{\zeta} = \zeta\left(\overset{\star}{x}\right)$ and the optimal radius $R^\star = \sqrt{\overset{\star}{\zeta}}$.
In order to approximate a global search procedure we follow the author in [9] who considered that the optimal $\mathcal{J}^{min}(\overset{\star}{x})$ is valid only at sampling points a where $g(x) \leq 0$ and $H(x) > 0$. Henc we present the following algorithm.

(i) Solve the direct problems (6) and (7) in the unperturbed domain ($\omega_\varepsilon = \emptyset$).
(ii) Compute the first and second topological derivatives G defined in (27) and (33)
(iii) Solve the system(1) on the sampling set

$$\mathcal{A} = \{x \in \Omega, g(x) \leq 0 \text{ and } H(x) > 0\}$$

(iv) Determine the point $\overset{\star}{x}$ minimum of $\zeta(x) = -\dfrac{g(x)}{H(x)}$.

Remark 2. In the following examples we can choose conductivities and the flux as polynomials with low degrees.
The reconstructed radius and center depend essentianly of conductivities values.
This reconstruction is available for a large choice of values to enable the detection of a general global minimum more than to a local one.

4.1 Example

We first present results of four inclusions with the same size but situated at different distance from the center and the boundary of the domain and with the same with radius $r_1 = 0.2$ and $r_2 = 0.1$.
In this example we choose the flux $g_3(x, y) = x^3 + y^3$ and almost the same conductivities $\sigma_0 = 12, \sigma_1 = 9.6, \sigma_2 = 19.2$. In this example we vanished the source term to observe its impact on the shape reconstruction.
We aim to get the best location where our method attempt the best detection of the inclusion.
Figure 2 represents the distribution the topological of order one and the topological derivative of order two for inclusion centred $(0.5, 0.5)$.
We present in (4.1) the error of reconstruction of the center and the radius.

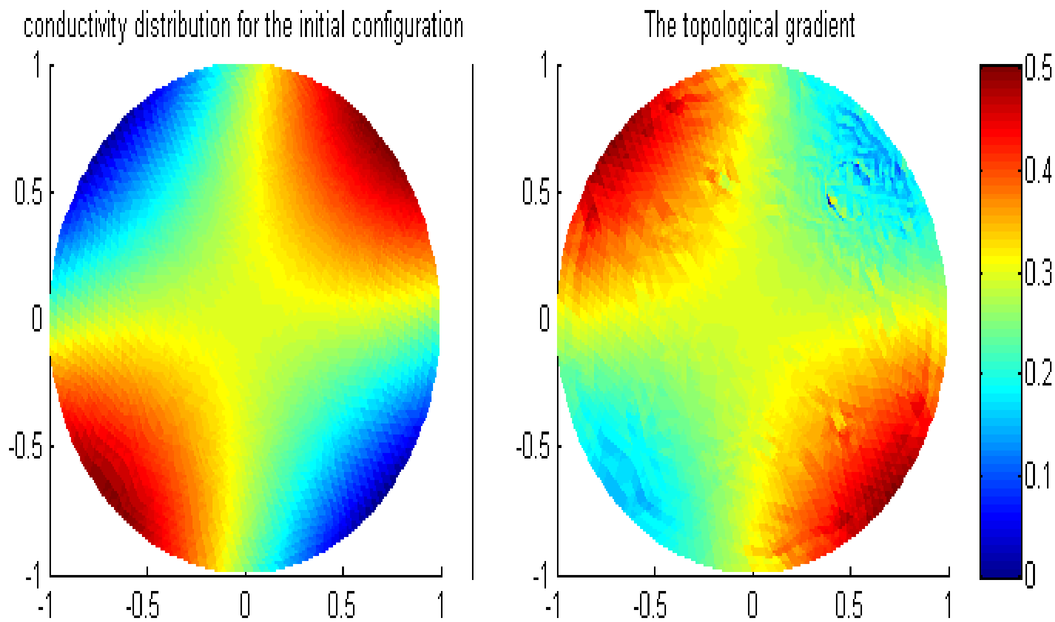

Fig. 2. On the left the first order expansion of the topological derivative and on the right the second order expansion with respcet to an inclusion centered at $(0.5, 0.5)$.

Parameters	g	r_1	r_{11}	$x-$center	$y-$center	J	Error
Real	x^3+y^3	0.2	0.1	0.5	0.5	$5.34e^{-22}$	0 %
Estimated	x^3+y^3	0.1999	0.0999	0.4	0.46	$4.91e^{-22}$	5.38%

Table 1. Table of the reconstruction of the center and the radius of the coated inclusions.

Remark 3. In the last exemple we remark that in the abscence of source term the reconstruction remains possible. The reconstruction is possible when the second order derivative is positive almost everywhere[9] to allow the search of the minimum everywhere and without restriction.

5 Conclusion

In this paper, we have developed the asymptotic expansion of cost functionals with respect to the insertion of small coated inclusion in order to minimize an energy-like functional. The implementation of the obtained first order topological derivative in a numerical algorithm doesn't lead to an interesting location of the shape, so we pushed the asymptotic expansion to the second order. We expressed the radius in fonction of the first and the second order topological derivative and we implemented it in a numerical algorithm which is effective and able to approximate the localisation and the size of the unknown coated inclusions.
A necessary condition of the existence of the global minimimum is the positivity of second order topological derivative but numerical tests show that this derivative is mostly negative on the outline of the inclusion. We can start from the detection of zones where the existence of the minumum is not insured and which correspond a priori to the outline of the inlcusion and then we can have a good approximation of the center and the radius. Certainly this result lacks theoretical justification but it remains an interesting challenge.

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