# Mathematical analysis of a two-stage Anaerobic Digestion model with production of hydrogen and methane

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**RÉSUMÉ.** La digestion anaérobie est un processus naturel de production de biogaz à partir de déchets organiques, en absence d'oxygène. Ce gaz peut être utilisé comme source d'énergie, d'où l'intérêt croissant aux procédés permettant l'optimisation de la production du biogaz. Ce travail présente un modèle mathématique correspondant à un processus de production de biogaz en deux phases, stimulant l'activité de la biomasse pour produire le maximum de méthane et d'hydrogène. Les deux phases du processus se déroulent dans deux bioréacteurs différents et sont décrites par deux systèmes avec un nombre égal d'équations. Nous nous proposons de faire l'analyse mathématique de ces systèmes. Pour cela, nous déterminons les points d'équilibre et donnons les conditions de leur existence et de leur stabilité locale. Les résultats obtenus sont illustrés par des simulations et des diagrammes opératoires permettant de bien comprendre le processus.

**ABSTRACT.** The Anaerobic Digestion (AD) is a natural process of biogas production from organic wastes. This gas can be used as a source of energy, hence the growing interest in processes allowing the optimization of biogas production. This work deals with the mathematical analysis of a continuous process model of AD in a cascade of two different bioreactors. The aim is to stimulate the activity of biomass, in order to produce a maximum rate of methane and hydrogen. The reactions are described by two systems with an equal number of equations in both bioreactors. We determine the equilibrium points of these systems and we give necessary and sufficient conditions for their existence and stability. Our results are illustrated by numerical simulations and operatory diagrams for the well understanding of the process.

**MOTS-CLÉS :** Digestion Anaérobie, Biogaz, Point d'équilibre, Stabilité locale, Diagrammes opératoires

**KEYWORDS :** Anaerobic digestion, Biogas, Equilibrium points, Local stability, Operating diagrams.

## 1. Introduction

The Anaerobic Digestion is a biotechnological process of organic material degradation with $H_2$ production as a non-accumulating intermediate product. This multi-step process presents an interest to the scientific and industrial community since hydrogen can be used as a substrate for several reactions, in addition to its consumption by hydrogenotrophic methanogens to produce $CH_4$ and $CO_2$. The produced biogas can be used as a source of energy. However, this biodegradation is a slow reaction, that's why a process based on connected bioreactors has been suggested in [2,3].
The AD process has been described in the literature by different models, see for example [1] and [4] where the complete anaerobic digestion model (ADM1) is presented. This model is very complex and uses many groups of microorganisms, involved in a complex production process of methane and carbon dioxide. The ADM1 model is based on many equations with a large number of variables and parameters. A lot of simplified models exist in the literature, see for example [5] and [6] and the references therein. In [2] and [3], the authors present a mathematical model of the AD process taking place in two connected bioreactors, operating in continuous mode (the input flow is equal to the output flow) in order to optimize the production of hydrogen and methane.
The main objective of this study is to analyze the mathematical model of [3], that describes an AD process in a cascade of two bioreactors, with production of hydrogen and three intermediate products (acetate, propionate and butyrate) in the first bioreactor ($BR_1$) and of methane in the second one ($BR_2$).

## 2. The mathematical models

In the bioreactor $BR_1$, the fast growing acidogens and $H_2$-producing microorganisms are developed in a volume $V_1$. These products are involved in the production of acetate, propionate, butyrate and $H_2$ (acidogenesis). Meanwhile, the slow growing acetogens and methanogens are developed in the second-stage methanogenic bioreactor $BR_2$, with working volume $V_2$ and in which the produced propionate and butyrate are turned to acetate (acetogenesis) and finally to $CH_4$ and $CO_2$ (methanogenesis), [3], as shown in Figure1.

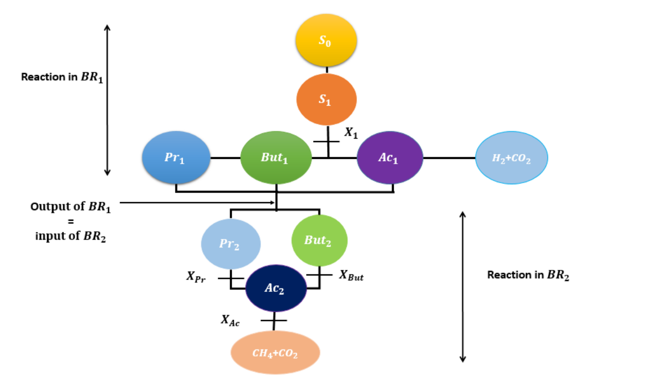


**Figure 1.** *The reactions in both bioreactors $BR_1$ and $BR_2$.*

The reactions in $\mathrm{BR}_1$ are described by a system of six differential equations in which the biomass with concentration $X_1$ consumes the effluent substrate $(S_0)$, which is transformed after hydrolysis in $S_1$, in order to produce $H_2$ and three intermediate products: the propionate $(Pr_1)$, the butyrate $(But_1)$ and the acetate $(Ac_1)$.

The model is written as follows:

$$\begin{cases} \dot{S_0} = -D_1 S_0 - \beta X_1 S_0 + D_1 Y_P S_0^{in} \\ \dot{S_1} = -D_1 S_1 + \beta X_1 S_0 - \frac{\mu_1(S_1)}{Y_1} X_1 \\ \dot{X_1} = (\mu_1(S_1) - D_1) X_1 \\ \dot{Pr_1} = \frac{\mu_1(S_1)}{Y_{Pr_1}} X_1 - D_1 Pr_1 \\ \dot{But_1} = \frac{\mu_1(S_1)}{Y_{But_1}} X_1 - D_1 But_1 \\ \dot{Ac_1} = \frac{\mu_1(S_1)}{Y_{Ac_1}} X_1 - D_1 Ac_1. \end{cases} \quad [2.1]$$

where $S_0^{in}$ is the inlet substrate's concentration in $\mathrm{BR}_1$, $D_1$ is the dilution rate in $\mathrm{BR}_1$, $Y_\alpha$, $\alpha = 1, Pr_1, But_1, Ac_1$, are yield coefficients. $\mu_1(.)$ is the growth function of acidogenic bacteria.
We assume that the growth function $\mu_1(.)$ satisfies the following condition:

$(H_1)$ $\mu_1(0) = 0$ and for all $S_1 > 0$, $\mu_1'(S_1) > 0$ .

Hypothesis $(H_1)$ means that without substrate there is no growth and that the growth rate of the biomass $X_1$ increases with the concentration of the substrate $S_1$. We denote by $\lambda_1$ the solution of the equation $\mu_1(S_1) = D_1$, if it exists. Otherwise, $\lambda_1 = +\infty$.

In the other hand, the methane is produced in the second bioreactor $\mathrm{BR}_2$, on the base of the acetate, propionate and butyrate produced in $\mathrm{BR}_1$. The following system of six differential equations describes the consumption of the propionate, butyrate and acetate substrates ($Pr_2$ , $But_2$ and $Ac_2$ , respectively) by the corresponding bacteria with concentration $X_{Pr}$ $X_{But}$ and $X_{Ac}$ respectively. The model is written as follows:

$$\begin{cases} \dot{Pr_2} = D_2(Pr_1 - Pr_2) - \frac{\mu_{Pr}(Pr_2)}{Y_{Pr_2}} X_{Pr} \\ \dot{X_{Pr}} = (\mu_{Pr}(Pr_2) - D_2) X_{Pr} \\ \dot{But_2} = D_2(But_1 - But_2) - \frac{\mu_{But}(But_2)}{Y_{But_2}} X_{But} \\ \dot{X_{But}} = (\mu_{But}(But_2) - D_2) X_{But} \\ \dot{Ac_2} = D_2(Ac_1 - Ac_2) + \frac{\mu_{Pr}(Pr_2)}{Y_{Pr_2}} X_{Pr} + \frac{\mu_{But}(But_2)}{Y_{But_2}} X_{But} - \frac{\mu_{Ac}(Ac_2)}{Y_{Ac_2}} X_{Ac} \\ \dot{X_{Ac}} = (\mu_{Ac}(Ac_2) - D_2) X_{Ac}. \end{cases} \quad [2.2]$$

where $Pr_1$, $But_1$ and $Ac_1$ are the inlet substrates concentrations in $\mathrm{BR}_2$, from $\mathrm{BR}_1$, $D_2$ is the dilution rate in $\mathrm{BR}_2$, $Y_\theta$, for $\theta = Pr_2, But_2, Ac_2$ are yield coefficients and $\mu_{Pr}(.)$, $\mu_{But}(.)$ and $\mu_{Ac}(.)$ are the specific growth function of $X_{Pr}$ $X_{But}$ and $X_{Ac}$ on $Pr_2$ , $But_2$ and $Ac_2$ , respectively.
We assume that:

$(H_2)$ $\mu_{Pr}(0) = 0$ and for all $S > 0$, $\mu'_{Pr}(S) > 0$.

$(H_3)$ $\mu_{But}(0) = 0$ and for all $S > 0$, $\mu'_{But}(S) > 0$.

$(H_4)$ $\mu_{Ac}(0) = 0$ and for all $S > 0$, $\mu'_{Ac}(S) > 0$.

For $A = Pr, But, Ac$, we denote by $\lambda_A$ the solution of the equation $\mu_A(S) = D_2$, if it exists. Otherwise, $\lambda_A = +\infty$.

## 3. Analysis of the models

In this section, we first give two preliminary results, then we determine the equilibrium points of each mathematical model in the bioreactors $\text{BR}_1$ and $\text{BR}_2$, as well as the conditions of their existence and stability.

**Proposition 3.1** *For any nonnegative initial conditions, the solutions of [2.1] remain nonnegative and are bounded. Moreover, the set*
$\Omega_1 = \left\{(S_0, S_1, X_1, Pr_1, But_1, Ac_1) \in I\!R^6_+ : Y_1 S_0(t) + Y_1 S_1(t) + \frac{1}{2} X_1(t) + \frac{1}{6}(Y_{Pr_1} Pr_1(t) + Y_{But_1} But_1(t) + Y_{Ac_1} Ac_1(t)) = Y_1 Y_P S_0^{in}\right\}$
*is invariant and attractive.*

This result can be proven by standard arguments. For model [2.2], we have:

**Proposition 3.2** *For any nonnegative initial condition, the solutions of [2.2] remain nonnegative and are bounded. Moreover, the set*
$\Omega_2 = \left\{(Pr_2, X_{Pr}, But_2, X_{But}, Ac_2, X_{Ac}) \in I\!R^6_+ : 2Pr_2(t) + 2But_2(t) + Ac_2(t) + \frac{X_{Pr}(t)}{Y_{Pr_2}} + \frac{X_{But}(t)}{Y_{But_2}} + \frac{X_{Ac}(t)}{Y_{Ac_2}} = 2Pr_1 + 2But_1 + Ac_1\right\}$
*is invariant and attractive.*

### 3.1. Equilibria of the mathematical model in $\text{BR}_1$

The study of the equilibria of model [2.1] proves the existence of three equilibria: a washout equilibrium which always exists and two positive equilibria. The description of the equilibria is given in the following result:

**Proposition 3.3** *The system [2.1] admits at most three equilibrium points, given by:*

*– The washout equilibrium* $E_0 = (Y_P S_0^{in}, 0, 0, 0, 0, 0)$

*– The positive equilibria* $E_{1i} = (S^*_{0i}, \lambda_1, X^*_{1i}, \frac{X^*_{1i}}{Y_{Pr_1}}, \frac{X^*_{1i}}{Y_{But_1}}, \frac{X^*_{1i}}{Y_{Ac_1}})$*, for* $i = 1, 2$*, where* $S^*_{0i}$ *are the solutions of* $\beta Y_1 S_0^2 - S_0\left(\beta Y_1 (Y_P S_0^{in} - \lambda_1) + D_1\right) + D_1 Y_P S_0^{in} = 0$*, when they exist and* $X^*_{1i} = Y_1(Y_P S_0^{in} - \lambda_1 - S^*_{0i})$*, for* $i = 1, 2$.

**Proof** *By solving the following algebraic system*

$$\begin{cases} -D_1 S_0 - \beta X_1 S_0 + D_1 Y_P S_0^{in} = 0 \\ -D_1 S_1 + \beta X_1 S_0 - \frac{\mu_1(S_1)}{Y_1} X_1 = 0 \\ (\mu_1(S_1) - D_1) X_1 = 0 \\ \frac{\mu_1(S_1)}{Y_{Pr_1}} X_1 - D_1 Pr_1 = 0 \\ \frac{\mu_1(S_1)}{Y_{But_1}} X_1 - D_1 But_1 = 0 \\ \frac{\mu_1(S_1)}{Y_{Ac_1}} X_1 - D_1 Ac_1 = 0. \end{cases} \qquad [3.3]$$

*The third equation of [3.3] implies that* $X_1 = 0$ *or* $\mu_1(S_1) = D_1$.
*If* $X_1 = 0$ *then* $S_0 = Y_P S_0^{in}$ *and* $S_1 = Pr_1 = But_1 = Ac_1 = 0$*. Thus, the washout*

*equilibrium* $E_0 = (Y_P S_0^{in}, 0, 0, 0, 0, 0)$ *always exists.*
*Otherwise,* $X_1 > 0$ *and* $\mu_1(S_1) = D_1$*. The sum of the first and the second equation of [3.3] gives*

$$D_1(Y_P S_0^{in} - S_0 - \lambda_1 - \frac{X_1}{Y_1}) = 0.$$

*We obtain that* $X_1 = Y_1(Y_P S_0^{in} - S_0 - \lambda_1)$*. Replacing* $X_1$ *in the first equation leads to the following second order equation*

$$\beta Y_1 S_0^2 - S_0 \left(\beta Y_1(Y_P S_0^{in} - \lambda_1) + D_1\right) + D_1 Y_P S_0^{in} = 0 \qquad [3.4]$$

*If the discriminant of [3.4] is negative, then there is no real solutions to [3.4]. So, the system [2.2] has no positive equilibria.*
*Otherwise, [3.4] admits one or two different solutions. If the solutions are nonnegative, system [2.2] admits one or two positive equilibria.*

The conditions of existence of equilibrium points of [2.1] are given with respect to the control parameters which are the inflowing substrate $S_0^{in}$ and the dilution $D_1$. The local stability conditions are obtained by linearization and calculation of the Jacobian matrix at the equilibrium points.

**Proposition 3.4** *The conditions of existence and stability of equilibria of [2.1] are given in Table 1:*

| Equilibria | Existence conditions | Stability conditions |
|---|---|---|
| $E_0$ | always exists | always stable |
| $E_{11}$ | $\Delta > 0$ and $S_0^{in} > \frac{1}{Y_P}(\lambda_1 + \frac{D_1}{\beta Y_1})$ | if it exists |
| $E_{12}$ | $\Delta > 0$ and $S_0^{in} > \frac{1}{Y_P}(\lambda_1 + \frac{D_1}{\beta Y_1})$ | unstable |

**Table 1.** *Existence and stability conditions of equilibria of model [2.1]*

*with* $\Delta = \left(\beta Y_1(Y_P S_0^{in} - \lambda_1) + D_1\right)^2 - 4\beta Y_1 Y_P D_1 S_0^{in}$ *is the discriminant of [3.4].*
*Furthermore, if* $\Delta = 0$*, the equilibrium* $E_{11}$ *is equal to the equilibrium* $E_{12}$*. It exists if* $S_0^{in} > \frac{1}{Y_P}(\lambda_1 + \frac{D_1}{\beta Y_1})$*.*

**Proof**

– *Existence conditions:*
*If* $\Delta > 0$*,* $S_{01}^*$ *and* $S_{02}^*$ *are the solutions of equation [3.4]. So, they verify:*

$$S_{01}^* S_{02}^* = \frac{D_1 Y_P}{\beta Y_1} S_0^{in} \quad \textit{and} \quad S_{01}^* + S_{02}^* = Y_P S_0^{in} - \lambda_1 + \frac{D_1}{\beta Y_1}$$

*Then,* $S_{01}^*$ *and* $S_{02}^*$ *are positive if, and only if,* $S_0^{in} > \frac{1}{Y_P}(\lambda_1 - \frac{D_1}{\beta Y_1})$*.*
*Since* $X_{1i} = Y_1(Y_P S_0^{in} - S_{0i} - \lambda_1)$*, for* $i = 1, 2$*, we obtain:*

$$X_{11}^* X_{12}^* = \frac{D_1 Y_1}{\beta} \lambda_1 \quad \textit{and} \quad X_{11}^* + X_{12}^* = Y_1(Y_P S_0^{in} - \lambda_1 - \frac{D_1}{\beta Y_1})$$

*Thus,* $X_{11}^*$ *and* $X_{12}^*$ *are positive if, and only if,* $S_0^{in} > \frac{1}{Y_P}(\lambda_1 + \frac{D_1}{\beta Y_1})$*. If this condition is verified, it implies that* $S_0^{in} > \frac{1}{Y_P}(\lambda_1 - \frac{D_1}{\beta Y_1})$*, hence* $S_{01}^*$ *and* $S_{02}^*$ *are positive.*

*In the case* $\Delta = 0$*, equation [3.4] admits a unique solution* $S_0^* = \frac{Y_P S_0^{in} - \lambda_1}{2} + \frac{D_1}{2\beta Y_1}$*, thereby,* $X_1^* = Y_1(\frac{Y_P S_0^{in} - \lambda_1}{2} - \frac{D_1}{2\beta Y_1})$ *which is positive if* $S_0^{in} > \frac{1}{Y_P}(\lambda_1 + \frac{D_1}{\beta Y_1})$ *which means that* $S_0^{in} > \frac{1}{Y_P}(\lambda_1 - \frac{D_1}{\beta Y_1})$*, hence* $S_0^*$ *is positive.*

*– Stability conditions:*

*Let* $J$ *be the Jacobian matrix of [2.1] at an equilibrium* $(S_0, S_1, X_1, Pr_1, But_1, Ac_1)$*.* $J$ *is a block diagonal matrix given by* $J := \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array}\right)$ *with*

$$A := \begin{pmatrix} -D_1 - \beta X_1 & 0 & -\beta S_0 \\ \beta X_1 & -D_1 - \frac{\mu_1'(S_1)}{Y_1} X_1 & \beta S_0 - \frac{\mu_1(S_1)}{Y_1} \\ 0 & \mu_1'(S_1) X_1 & \mu_1(S_1) - D_1 \end{pmatrix},$$

$$B := \begin{pmatrix} 0 & \frac{\mu_1'(S_1)}{Y_{Pr_1}} X_1 & \frac{\mu_1(S_1)}{Y_{Pr_1}} \\ 0 & \frac{\mu_1'(S_1)}{Y_{But_1}} X_1 & \frac{\mu_1(S_1)}{Y_{But_1}} \\ 0 & \frac{\mu_1'(S_1)}{Y_{Ac_1}} X_1 & \frac{\mu_1(S_1)}{Y_{Ac_1}} \end{pmatrix} \text{ and } C := \begin{pmatrix} -D_1 & 0 & 0 \\ 0 & -D_1 & 0 \\ 0 & 0 & -D_1 \end{pmatrix}.$$

*The eigenvalues of* $J$ *are therefore the eigenvalues of* $A$ *and* $C$*. As* $C$ *is a diagonal matrix, then it admits three eigenvalues equal to* $-D_1 < 0$*. So, to determine the stability conditions of each equilibrium, the eigenvalues of the matrix* $A$ *must be determined.*

*At* $E_0 = (Y_P S_0^{in}, 0, 0, 0, 0, 0)$*, the matrix* $A_0 = \begin{pmatrix} -D_1 & 0 & -\beta s_0^{in} \\ 0 & -D_1 & \beta s_0^{in} \\ 0 & 0 & -D_1 \end{pmatrix}$ *admits three eigenvalues equal to* $-D_1 < 0$*. Hence,* $E_0$ *is locally asymptotically stable.*

*At* $E_{1i} = (S_{0i}^*, \lambda_1, X_{1i}^*, \frac{X_{1i}^*}{Y_{Pr_1}}, \frac{X_{1i}^*}{Y_{But_1}}, \frac{X_{1i}^*}{Y_{Ac_1}})$*, for* $i = 1, 2$*, the matrix* $A$ *writes*

$A_i := \begin{pmatrix} -D_1 - \beta X_{1i}^* & 0 & -\beta S_{0i}^* \\ \beta X_{1i}^* & -D_1 - \frac{\mu_1'(\lambda_1)}{Y_1} X_{1i}^* & \beta S_{0i}^* - \frac{D_1}{Y_1} \\ 0 & \mu_1'(\lambda_1) X_{1i}^* & 0 \end{pmatrix}$*. The characteristic polynomial of* $A_i$ *is given by:*

$P_i(\xi) = det(A_i - \xi I_3)$

$= -(D_1 + \xi)\left(\xi^2 + \xi(D_1 + \beta X_{1i}^* + \frac{\mu_1'(\lambda_1)}{Y_1} X_{1i}^*) + \frac{\mu_1'(\lambda_1)}{Y_1} X_{1i}^*(D_1 + \beta(X_{1i}^* - Y_1 S_{0i}^*))\right)$

*where* $I_3$ *is the* $3 \times 3$ *identity matrix. The polynomial of degree two in* $P_i(\xi)$ *admits two roots* $\xi_1$ *and* $\xi_2$ *such that*

$$\xi_1 + \xi_2 = -(D_1 + \beta X_{1i}^* + \frac{\mu_1'(\lambda_1)}{Y_1} X_{1i}^*) \quad and \quad \xi_1 \xi_2 = \frac{\mu_1'(\lambda_1)}{Y_1} X_{1i}^*(D_1 + \beta(X_{1i}^* - Y_1 S_{0i}^*))$$

$\xi_1$ *and* $\xi_2$ *are negative if and only if* $D_1 + \beta(X_{1i}^* - Y_1 S_{0i}^*) > 0$ *which is equivalent to* $S_{0i}^* < \frac{Y_P S_0^{in} - \lambda_1}{2} + \frac{D_1}{2 Y_1 \beta}$*, since* $X_{1i}^* = Y_1(Y_P S_0^{in} - S_{0i}^* - \lambda_1)$*.*

*Now, for* $\Delta > 0$*, as*

$S_{01}^* = \frac{\beta Y_1 (Y_P S_0^{in} - \lambda_1) + D_1 - \sqrt{\Delta}}{2 Y_1 \beta} < \frac{Y_P S_0^{in} - \lambda_1}{2} + \frac{D_1}{2 Y_1 \beta}$*, therefore,* $E_{11}$ *is stable when it exists.*

$S_{02}^{*} = \frac{\beta Y_1(Y_P S_0^{in} - \lambda_1) + D_1 + \sqrt{\Delta}}{2Y_1\beta} > \frac{Y_P S_0^{in} - \lambda_1}{2} + \frac{D_1}{2Y_1\beta}$, *therefore,* $E_{12}$ *is unstable.*

## 3.2. Equilibria of the mathematical model in BR$_2$

The equilibria of model [2.2] are the solutions of the following algebraic system:

$$\begin{cases} D_2(Pr_1 - Pr_2) - \frac{\mu_{Pr}(Pr_2)}{Y_{Pr_2}} X_{Pr} = 0 \\ (\mu_{Pr}(Pr_2) - D_2) X_{Pr} = 0 \\ D_2(But_1 - But_2) - \frac{\mu_{But}(But_2)}{Y_{But_2}} X_{But} = 0 \\ (\mu_{But}(But_2) - D_2) X_{But} = 0 \\ D_2(Ac_1 - Ac_2) + \frac{\mu_{Pr}(Pr_2)}{Y_{Pr_2}} X_{Pr} + \frac{\mu_{But}(But_2)}{Y_{But_2}} X_{But} - \frac{\mu_{Ac}(Ac_2)}{Y_{Ac_2}} X_{Ac} = 0 \\ (\mu_{Ac}(Ac_2) - D_2) X_{Ac} = 0 \end{cases} \quad [3.5]$$

**Proposition 3.5** *System [2.2] admits at most seventeen equilibrium points defined in Table 2.*

| Equilibria | $Pr_2$ | $X_{Pr}$ | $But_2$ | $X_{But}$ | $Ac_2$ | $X_{Ac}$ |
|---|---|---|---|---|---|---|
| $F_{00}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $F_{0i}$ | $a_1X_{1i}^*$ | 0 | $a_2X_{1i}^*$ | 0 | $a_3X_{1i}^*$ | 0 |
| $F_{1i}$ | $a_1X_{1i}^*$ | 0 | $a_2X_{1i}^*$ | 0 | $\lambda_{Ac}$ | $Y_{Ac_2}\bar{Ac}_i$ |
| $F_{2i}$ | $a_1X_{1i}^*$ | 0 | $\lambda_{But}$ | $Y_{But_2}\bar{But}_i$ | $a_3X_{1i}^* + \bar{But}_i$ | 0 |
| $F_{3i}$ | $a_1X_{1i}^*$ | 0 | $\lambda_{But}$ | $Y_{But_2}\bar{But}_i$ | $\lambda_{Ac}$ | $Y_{Ac_2}(\bar{Ac}_i + \bar{But}_i)$ |
| $F_{4i}$ | $\lambda_{Pr}$ | $Y_{Pr_2}\bar{Pr}_i$ | $a_2X_{1i}^*$ | 0 | $a_3X_{1i}^* + \bar{Pr}_i$ | 0 |
| $F_{5i}$ | $\lambda_{Pr}$ | $Y_{Pr_2}\bar{Pr}_i$ | $a_2X_{1i}^*$ | 0 | $\lambda_{Ac}$ | $Y_{Ac_2}(\bar{Ac}_i + \bar{Pr}_i)$ |
| $F_{6i}$ | $\lambda_{Pr}$ | $Y_{Pr_2}\bar{Pr}_i$ | $\lambda_{But}$ | $Y_{But_2}\bar{But}_i$ | $a_3X_{1i}^* + \bar{Pr}_i + \bar{But}_i$ | 0 |
| $F_{7i}$ | $\lambda_{Pr}$ | $Y_{Pr_2}\bar{Pr}_i$ | $\lambda_{But}$ | $Y_{But_2}\bar{But}_i$ | $\lambda_{Ac}$ | $Y_{Ac_2}(\bar{Ac}_i + \bar{Pr}_i + \bar{But}_i)$ |

**Table 2.** *Equilibrium points of [2.2]*

*with*

$$a_1 = \frac{1}{Y_{Pr_1}} \ , \ a_2 = \frac{1}{Y_{But_1}} \ , \ a_3 = \frac{1}{Y_{Ac_1}}$$

$$\bar{Pr}_i = a_1X_{1i}^* - \lambda_{Pr} \ , \ \bar{But}_i = a_2X_{1i}^* - \lambda_{But} \ , \ \bar{Ac}_i = a_3X_{1i}^* - \lambda_{Ac}.$$

**Proof** *The second equation of [3.5] gives $X_{Pr} = 0$ or else $\mu_{Pr}(Pr_2) = D_2$.*
*If $X_{Pr} = 0$, then the first equation of [3.5] gives $Pr_2 = Pr_1$.*
*On the other hand, we know that according to the Proposition 3.3, we have $Pr_1 = 0$ or $Pr_{1i} = \frac{X_{1i}^*}{Y_{Pr_1}}$, for $i = 1, 2$. Then $Pr_2 = 0$ or $Pr_{2i} = a_1X_{1i}^*$ , for $i = 1, 2$, with $a_1 = \frac{1}{Y_{Pr_1}}$.*
*Otherwise, we have $X_{Pr} > 0$ and $Pr_2 = \lambda_{Pr}$ with $\lambda_{Pr}$ is a solution of $\mu_{Pr}(Pr_2) = D_2$. Then, the first equation of [3.5] gives $X_{Pr} = Y_{Pr_2}(Pr_1 - \lambda_{Pr})$. We replace $Pr_1$ by its value to get $X_{Pr,i} = Y_{Pr_2}(a_1X_{1i}^* - \lambda_{Pr})$, for $i = 1, 2$. Let $\bar{Pr}_i = a_1X_{1i}^* - \lambda_{Pr}$. Therefore, $X_{Pr,i} = Y_{Pr_2}\bar{Pr}_i$, for $i = 1, 2$.*
*The case where $Pr_1 = 0$ is to be rejected, otherwise $X_{Pr}$ becomes negative.*
*In conclusion, for $i = 1, 2$, we can have $(Pr_2, X_{Pr}) = (0, 0)$ or $(Pr_{2i}, X_{Pr}) = (a_1X_{1i}^*, 0)$ or $(Pr_{2i}, X_{Pr,i}) = (\lambda_{Pr}, Y_{Pr_2}\bar{Pr}_i)$.*
*Using the third and the fourth equation of [3.5], we show in the same way that, for $i = 1, 2$, $(But_2, X_{But}) = (0, 0)$ or $(But_{2i}, X_{But}) = (a_2X_{1i}^*, 0)$ or else $(But_{2i}, X_{But,i}) = (\lambda_{But}, Y_{But_2}\bar{But}_i)$, with $a_2 = \frac{1}{Y_{But_1}}$ and $\bar{But}_i = a_2X_{1i}^* - \lambda_{But}$.*
*Now, the sixth equation of [3.5] gives $X_{Ac} = 0$ or else $\mu_{Ac}(Ac_2) = D_2$.*
*If $X_{Ac} = 0$, the fifth equation of [3.5] becomes*

$$D_2(Ac_1 - Ac_2) + \frac{\mu_{Pr}(Pr_2)}{Y_{Pr_2}}X_{Pr} + \frac{\mu_{But}(But_2)}{Y_{But_2}}X_{But} = 0.$$

*Note that:*

*– If $X_{Pr} = X_{But} = 0$, then $Ac_2 = Ac_1$. As $Ac_1 = 0$ or $Ac_{1i} = \frac{X_{1i}^*}{Y_{Ac_1}}$, for $i = 1, 2$, according to Proposition 3.3, so $Ac_2 = 0$ or $Ac_{2i} = a_3X_{1i}^*$ with $a_3 = \frac{1}{Y_{Ac_1}}$.*

*– If $X_{Pr} = 0$ and $X_{But} \neq 0$, then $Ac_2 = Ac_1 + \frac{X_{But}}{Y_{But_2}}$. Thus,*

$$Ac_{2i} = a_3 X_{1i}^* + \bar{But}_i, \ for \ i = 1, 2.$$

*– If $X_{Pr} \neq 0$ and $X_{But} = 0$, then $Ac_2 = Ac_1 + \frac{X_{Pr}}{Y_{Pr_2}}$. Thus,*

$$Ac_{2i} = a_3 X_{1i}^* + \bar{Pr}_i, \ for \ i = 1, 2.$$

*– If $X_{Pr} \neq 0$ and $X_{But} \neq 0$, then $Ac_2 = Ac_1 + \frac{X_{Pr}}{Y_{Pr_2}} + \frac{X_{But}}{Y_{But_2}}$. Thus,*

$$Ac_{2i} = a_3 X_{1i}^* + \bar{Pr}_i + \bar{But}_i, \ for \ i = 1, 2.$$

*Otherwise, we have $X_{Ac} > 0$ and $Ac_2 = \lambda_{Ac}$ with $\lambda_{Ac}$ is a solution of $\mu_{Ac}(Ac_2) = D_2$. Then, the fifth equation of [3.5] gives*

$$X_{Ac} = Y_{Ac_2}\left(Ac_1 - \lambda_{Ac} + \frac{1}{D_2}(\frac{\mu_{Pr}(Pr_2)}{Y_{Pr_2}} X_{Pr} + \frac{\mu_{But}(But_2)}{Y_{But_2}} X_{But})\right)$$

*Note that:*

*– If $X_{Pr} = X_{But} = 0$, then $X_{Ac} = Y_{Ac_2}(Ac_1 - \lambda_{Ac})$. Thus,*

$$X_{Ac,i} = Y_{Ac_2}(a_3 X_{1i}^* - \lambda_{Ac}) = Y_{Ac_2}\bar{Ac}_i, with \ \bar{Ac}_i = a_3 X_{1i}^* - \lambda_{Ac}, \ for \ i = 1, 2.$$

*– If $X_{Pr} = 0$ and $X_{But} \neq 0$, then $X_{Ac} = Y_{Ac_2}(Ac_1 - \lambda_{Ac} + \frac{X_{But}}{Y_{But_2}})$. Thus,*

$$X_{Ac,i} = Y_{Ac_2}(\bar{Ac}_i + \bar{But}_i), \ for \ i = 1, 2.$$

*– If $X_{Pr} \neq 0$ and $X_{But} = 0$, then $X_{Ac} = Y_{Ac_2}(Ac_1 - \lambda_{Ac} + \frac{X_{Pr}}{Y_{Pr_2}})$. Thus,*

$$X_{Ac,i} = Y_{Ac_2}(\bar{Ac}_i + \bar{Pr}_i), \ for \ i = 1, 2.$$

*– If $X_{Pr} \neq 0$ and $X_{But} \neq 0$, then $X_{Ac} = Y_{Ac_2}(Ac_1 - \lambda_{Ac} + \frac{X_{Pr}}{Y_{Pr_2}} + \frac{X_{But}}{Y_{But_2}})$. Thus,*

$$X_{Ac,i} = Y_{Ac_2}(\bar{Ac}_i + \bar{Pr}_i + \bar{But}_i), \ for \ i = 1, 2.$$

We give the conditions for existence and stability of the equilibrium points of system [2.2], in the Appendix. As previously, the results are obtained by determining the sign of the real part of the eigenvalues of the Jacobian matrix at each equilibrium.

## 4. Conclusion

We are interested in this work in an AD process in two connected bioreactors. The different phases take place successively in the two bioreactors. The biological reactions in the first one allow the production of hydrogen while methane is produced at the exit of the second. Our work was devoted to an analytical study of the models of [3], where only a numerical study was carried out. We have studied the equilibria of the dynamic systems in each of the bioreactors. The study showed the existence of at most three equilibria in the first model while in the second, the system can have up to eight different types of equilibria. We determined the conditions of existence and stability according to the control parameters. We proved that, in $BR_1$, the washout equilibrium is always stable and that two positive equilibria can exist but one of them is stable whenever it exists, while the second is unstable. In $BR_2$, we showed in particular that the washout equilibrium is always stable too and two positive equilibria can exist and are stable whenever they exist. Thus, the system can exhibit a tri-stability behavior, depending on the control parameters. This careful study permits to give operating diagrams describing the behavior of the system. It makes it possible to determine what are the conditions on the dilutions and the inflowing concentrations so that the digestion process in the two bioreactors is that expected. These results can be used by biologists to optimize production rates of biogas.

## 5. Bibliographie


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## Appendix A

First, we define the following functions:

– $\varphi_1(D_2) = \beta Y_1 \lambda + D_1 + 2\beta Y_{Pr_1} \lambda_{Pr}$

– $\varphi_2(D_2) = \beta Y_1 \lambda + D_1 + 2\beta Y_{But_1} \lambda_{But}$

– $\varphi_3(D_2) = \beta Y_1 \lambda + D_1 + 2\beta Y_{Ac_1} \lambda_{Ac}$

– $\varphi_4(D_2) = \beta Y_1 \lambda + D_1 + 2\beta \frac{Y_{Ac_1} Y_{But_1}}{Y_{Ac_1} + Y_{But_1}} \lambda_{But}$

– $\varphi_5(D_2) = \beta Y_1 \lambda + D_1 + 2\beta \frac{Y_{Ac_1} Y_{But_1}}{Y_{Ac_1} + Y_{But_1}} (\lambda_{But} + \lambda_{Ac})$

– $\varphi_6(D_2) = \beta Y_1 \lambda + D_1 + 2\beta \frac{Y_{Ac_1} Y_{Pr_1}}{Y_{Ac_1} + Y_{Pr_1}} \lambda_{Pr}$

– $\varphi_7(D_2) = \beta Y_1 \lambda + D_1 + 2\beta \frac{Y_{Ac_1} Y_{Pr_1}}{Y_{Ac_1} + Y_{Pr_1}} (\lambda_{Pr} + \lambda_{Ac})$

– $\varphi_8(D_2) = \beta Y_1 \lambda + D_1 + 2\beta \frac{Y_{Ac_1} Y_{But_1} Y_{Pr_1}}{Y_{Ac_1} Y_{Pr_1} + Y_{Pr_1} Y_{But_1} + Y_{But_1} Y_{Ac_1}} (\lambda_{But} + \lambda_{Pr})$

| Equilibria | Existence conditions | Stability conditions |
|---|---|---|
| $F_{00}$ | always exists | always stable |
| $F_{01}$ | always exists | $S_0^{in} < g_1(D_2)$ and $S_0^{in} < f_1(D_2)$<br>$S_0^{in} < g_2(D_2)$ and $S_0^{in} < f_2(D_2)$<br>$S_0^{in} < g_3(D_2)$ and $S_0^{in} < f_3(D_2)$ |
| $F_{02}$ | always exists | $S_0^{in} < f_1(D_2)$ or [ $S_0^{in} > g_1(D_2)$ and $S_0^{in} > f_1(D_2)$]<br>$S_0^{in} < f_2(D_2)$ or [ $S_0^{in} > g_2(D_2)$ and $S_0^{in} > f_2(D_2)$]<br>$S_0^{in} < f_3(D_2)$ or [ $S_0^{in} > g_3(D_2)$ and $S_0^{in} > f_3(D_2)$] |
| $F_{11}$ | $S_0^{in} > f_3(D_2)$ or<br>[ $S_0^{in} > g_3(D_2)$ and $S_0^{in} < f_3(D_2)$] | $S_0^{in} < g_1(D_2)$ and $S_0^{in} < f_1(D_2)$<br>$S_0^{in} < g_2(D_2)$ and $S_0^{in} < f_2(D_2)$ |
| $F_{12}$ | $S_0^{in} < g_3(D_2)$ and $S_0^{in} > f_3(D_2)$ | $S_0^{in} < f_1(D_2)$ or [ $S_0^{in} > g_1(D_2)$ and $S_0^{in} > f_1(D_2)$]<br>$S_0^{in} < f_2(D_2)$ or [ $S_0^{in} > g_2(D_2)$ and $S_0^{in} > f_2(D_2)$] |
| $F_{21}$ | $S_0^{in} > \frac{\varphi_2(D_2)}{\gamma}$ or<br>[$S_0^{in} > g_2(D_2)$ and $S_0^{in} < f_2(D_2)$]<br>$S_0^{in} > f_4(D_2)$ or<br>[$S_0^{in} > g_4(D_2)$ and $S_0^{in} < f_4(D_2)$] | $S_0^{in} < g_1(D_2)$ and $S_0^{in} < f_1(D_2)$<br>$S_0^{in} < g_5(D_2)$ and $S_0^{in} < f_5(D_2)$ |
| $F_{22}$ | $S_0^{in} < g_2(D_2)$ and $S_0^{in} > f_2(D_2)$<br>$S_0^{in} < g_4(D_2)$ and $S_0^{in} > f_4(D_2)$ | $S_0^{in} < f_1(D_2)$ or [ $S_0^{in} > g_1(D_2)$ and $S_0^{in} > f_1(D_2)$]<br>$S_0^{in} < f_5(D_2)$ or [ $S_0^{in} > g_5(D_2)$ and $S_0^{in} > f_5(D_2)$] |
| $F_{31}$ | $S_0^{in} > f_2(D_2)$ or<br>[$S_0^{in} > g_2(D_2)$ and $S_0^{in} < f_2(D_2)$]<br>$S_0^{in} > f_5(D_2)$ or<br>[$S_0^{in} > g_5(D_2)$ and $S_0^{in} < f_5(D_2)$] | $S_0^{in} < g_1(D_2)$ and $S_0^{in} < f_1(D_2)$ |

– $\varphi_9(D_2) = \beta Y_1 \lambda + D_1 + 2\beta \frac{Y_{Ac_1} Y_{But_1} Y_{Pr_1}}{Y_{Ac_1} Y_{Pr_1} + Y_{Pr_1} Y_{But_1} + Y_{But_1} Y_{Ac_1}} (\lambda_{But} + \lambda_{Pr} + \lambda_{Ac})$

Let's define too $\gamma = \beta Y_1 Y_P$, $f_i(D_2) = \frac{\varphi_i(D_2)}{\gamma}$ and the functions

$$g_i(D_2) = \frac{\varphi_i(D_2)^2 - c^2}{2\gamma(c + \varphi_i(D_2) - 2D_1)}, i = 1, 2.$$

with $c = D_1 - \beta Y_1 \lambda$.

**Proposition 5.1** *The conditions of existence and stability of equilibria of [2.2] are presented in Table 3.*

| | | |
|---|---|---|
| $F_{32}$ | $S_0^{in} < g_2(D_2)$ and $S_0^{in} > f_2(D_2)$<br>$S_0^{in} < g_5(D_2)$ and $S_0^{in} > f_5(D_2)$ | $S_0^{in} < f_1(D_2)$ or [ $S_0^{in} > g_1(D_2)$ and $S_0^{in} > f_1(D_2)$ ] |
| $F_{41}$ | $S_0^{in} > f_1(D_2)$ or<br>[$S_0^{in} > g_1(D_2)$ and $S_0^{in} < f_1(D_2)$]<br>$S_0^{in} > f_6(D_2)$ or<br>[$S_0^{in} > g_6(D_2)$ and $S_0^{in} < f_6(D_2)$] | $S_0^{in} < g_2(D_2)$ and $S_0^{in} < f_2(D_2)$<br>$S_0^{in} < g_7(D_2)$ and $S_0^{in} < f_7(D_2)$ |
| $F_{42}$ | $S_0^{in} < g_1(D_2)$ and $S_0^{in} > f_1(D_2)$<br>$S_0^{in} < g_6(D_2)$ and $S_0^{in} > f_6(D_2)$ | $S_0^{in} < f_2(D_2)$ or [ $S_0^{in} > g_2(D_2)$ and $S_0^{in} > f_2(D_2)$]<br>$S_0^{in} < f_7(D_2)$ or [ $S_0^{in} > g_7(D_2)$ and $S_0^{in} > f_7(D_2)$] |
| $F_{51}$ | $S_0^{in} > f_1(D_2)$ or<br>[$S_0^{in} > g_1(D_2)$ and $S_0^{in} < f_1(D_2)$]<br>$S_0^{in} > f_7(D_2)$ or<br>[$S_0^{in} > g_7(D_2)$ and $S_0^{in} < f_7(D_2)$] | $S_0^{in} < g_2(D_2)$ and $S_0^{in} < f_2(D_2)$ |
| $F_{52}$ | $S_0^{in} < g_1(D_2)$ and $S_0^{in} > f_1(D_2)$<br>$S_0^{in} < g_7(D_2)$ and $S_0^{in} > f_7(D_2)$ | $S_0^{in} < f_2(D_2)$ or [ $S_0^{in} > g_2(D_2)$ and $S_0^{in} > f_2(D_2)$] |
| $F_{61}$ | $S_0^{in} > f_1(D_2)$ or<br>[ $S_0^{in} > g_1(D_2)$ and $S_0^{in} < f_1(D_2)$]<br>$S_0^{in} > f_2(D_2)$ or<br>[$S_0^{in} > g_2(D_2)$ and $S_0^{in} < f_2(D_2)$]<br>$S_0^{in} > f_8(D_2)$ or<br>[$S_0^{in} > g_8(D_2)$ and $S_0^{in} < f_8(D_2)$] | $S_0^{in} < g_9(D_2)$ and $S_0^{in} < f_9(D_2)$ |
| $F_{62}$ | $S_0^{in} < g_1(D_2)$ and $S_0^{in} > f_1(D_2)$<br>$S_0^{in} < g_2(D_2)$ and $S_0^{in} > f_2(D_2)$<br>$S_0^{in} < g_8(D_2)$ and $S_0^{in} > f_8(D_2)$ | $S_0^{in} < f_9(D_2)$ or [ $S_0^{in} > g_9(D_2)$ and $S_0^{in} > f_9(D_2)$ ] |
| $F_{71}$ | $S_0^{in} > f_1(D_2)$ or<br>[$S_0^{in} > g_1(D_2)$ and $S_0^{in} < f_1(D_2)$]<br>$S_0^{in} > f_2(D_2)$ or<br>[$S_0^{in} > g_2(D_2)$ and $S_0^{in} < f_2(D_2)$]<br>$S_0^{in} > f_9(D_2)$ or<br>[$S_0^{in} > g_9(D_2)$ and $S_0^{in} < f_9(D_2)$] | if it exists |
| $F_{72}$ | $S_0^{in} < g_1(D_2)$ and $S_0^{in} > f_1(D_2)$<br>$S_0^{in} < g_2(D_2)$ and $S_0^{in} > f_2(D_2)$<br>$S_0^{in} < g_9(D_2)$ and $S_0^{in} > f_9(D_2)$ | if it exists |

**Table 3.** *The existence and stability conditions of equilibria of system [2.2].*