Mathematical modelling and optimal control of banana black sigatoka disease

Franklin Platini Agouanet [a,*] — Jean Jules Tewa[b] — Magloire Remy Etoua[c]

[a] Department of Mathematics, University of Yaounde I,
PO Box 812 Yaounde, Cameroon
agouanetf@yahoo.com

[b] National Advanced School of Engineering
University of Yaounde I, Department of Mathematics and Physics
P.O. Box 8390 Yaounde, Cameroon
tewajules@gmail.com

[c] National Advanced School of Engineering of Yaounde
P.O. Box 8390 Yaounde, Cameroon

[*] Corresponding author
Tel.+(237) 693 22 98 42; agouanetf@yahoo.com

ABSTRACT. In this work, we propose a mathematical pathogen-host model which focused on the epidemiological cycle of banana black leaf streak disease and which takes into account control strategies to reduce the spores production ,this mainly thanks to an differential equations system with time delay in wich, the host dynamics is coupled to that of the fungus responsible for the disease. By using optimal control theory, we show that an optimal control exists for this problem and Pontryagin's maximum principle is used to characterize an optimal control. Numerical simulations are provided to illustrate our results.

RÉSUMÉ. Dans ce travail, nous proposons un modèle mathématiques hôte-pathogène axé sur le cycle épidémiologique de la maladie des raies noires du bananier et prenant en compte les stratégies de contrôle pour réduire la production des spores, ceci à partir d'un système d'équations différentielles avec retard dans lequel la dynamique de la plante hôte est couplé avec celle du champignon responsable de la maladie. En utilisant la théorie du contrôle optimal, nous montrons qu'un contrôle optimal existe pour ce problème et le principe du maximum de Pontryagin est utilisé pour caractériser ce contrôle optimal. Des simulations numériques sont faites pour illustrer nos résultats.

KEYWORDS : Mathematical modelling, black leaf streak disease, optimal control, sexual reproduction, asexual reproduction, delay

MOTS-CLÉS : Modélisation mathématiques, Maladie des raies noires, contrôle optimal, reproduction sexuée, reproduction asexuée, retard

1. Introduction

Banana is grown in more than 120 countries around the world, primarily in the tropical, rain-fed regions of Africa, Asia and Latin America, where there is very important in the food security of more than 400 million people and constitute a employment and income source for local populations[11]. However, both banana and banana-plantain cultures are hampered by plant parasitic nematodes, insect pests and foliar fungi(IRAD). Black sigatoka disease also called black leaf streak disease (BLSD) is caused by a pathogen fungi , *Mycosphaeralla fijiensis,* which is the most costly and damaging banana leaf disease worldwide([1]). BLSD causes significant decrease of photosynthetic surface by a general drying of the leaf system resulting the loss production of ranging from 20 to more than 50%[2].
Mycosphaerella fijiensis Morelet which reproduces sexually and asexually. Ascospores are the product of sexual reproduction, while conidia are produced through asexual reproduction and these spores are responsible for the spread of the disease. This fungus performs its entire life cycle on the banana tree and their disease cycle consists of four distinct stages that include spore germination, penetration of the host, symptom development and spore production.([1])
The susceptibility of the crop necessitates the cultural practices and use of multiple fungicides at relatively high frequencies. Such applications are potentially detrimental, not only to the environment, but also to those who live and work in areas in which banana is treated to control the disease([1]). In this paper, we propose a mathematical pathogen-host model which focused on the epidemiological cycle of the disease and which takes into account control strategies to reduce the spores production.

2. The model formulation

We formulate an optimal control pathogen host model for black sigatoka disease in order to derive optimal control strategies with minimal implementation cost. The control function, u represent time dependent efforts made to reduce the ascospores(sexual spore) and conidia(asexual spore) reproduction practiced on a time interval $[0, T]$ where T is the cropping season duration. Known practices of control efforts include cultural practices and treating by fungicides. We divide the host population(N) suppose constant into two compartments using variables S and I to describe the total number of susceptible leaf and infected leaf, respectively. On the other hand, Y and Z describe respectively the sexual and asexual spore groups in the pathogen population. Note that the term 'leaf' is a shorthand for the "leaf part that a lesion occupies", so that multiple infections cannot occur in the model.
In the Moungo region (Cameroon), the incubation times are the shorter (13-16 days) were obtained between May and November, a period more favorable for illness, while the longest was obtained in February (26 days)([6]), so the delay (τ) represents the incubation period of the disease. We assumed that, all the leaves are destroyed and the fungus also disappears at the end of cropping season, and that the spore dispersal rate is equal to the spore deposite rate[5].

let β and α the rate of deposit respectively sexual and asexual spore upon leaves, and $0 < p < 1$ be their infectivity(we assume that conidia infectivity is equal to the ascospores infectivity[9]), i.e. the probability that a spore in contact with a susceptible leaf succeeds to infect it. Let σ be the number of asexual spores produced per infected leaf per unit time, and γ be the number of sexual spores (ascospores) produced per infected leaf per unit time.

- Infestation function is given by $p(\beta X + \alpha Y)\dfrac{S}{N}$. Hence, use the fact that the density of susceptible leaves can be deduced from the infected leaf density through the delayed equation $s(t-\tau) = N - x(t-\tau)$, where $0 < \tau < T$ stands for the time delay, we have the dynamics of infected leaves given by:

$$\dot{I}(t) \quad = \quad p(\beta X(t) + \alpha Y(t))\left(1 - \frac{I(t-\tau)}{N}\right) - mI(t)$$

where m is the infectious leaves mortality rate.

- The asexual spore dynamics is given by $\dot{Y}(t) = \sigma I(t) - \beta Y(t)$ where $\sigma I(t)$ is the asexual spore production function.

- Sexual spore production thus results from the interaction between two individuals with compatible mating types ($+$ and $-$). However, sexual spore production is conditioned to the local presence of a mating partner, whose probability is $(I/2)/N$, assuming a balanced mating-type ratio and the sexual sopre production function is given by $\dfrac{\gamma}{2N}I^2$ [5]; hence the sexual spores dynamics is given by $\dot{X}(t) = \dfrac{\gamma}{2N}I(t)^2 - \beta X(t)$.

Hence, our model is given for $t \in [0, T]$ by: $\begin{cases} \dot{X} & = & p(\beta Y + \alpha Z)\left(1 - \dfrac{X(t-\tau)}{N}\right) - mX \\ \dot{X} & = & (1-u)\dfrac{\gamma}{2N}X^2 - \beta Y \\ \dot{Z} & = & (1-u)\sigma X - \alpha Z \end{cases}$

Let us use the following variable change: $x = \dfrac{I}{N}$, $y = \dfrac{Y}{N}$ and $z = \dfrac{Z}{N}$, so for $t \in [0, T]$ our model is gigen by:

$$\begin{cases} \dot{x} & = & p(\beta y + \alpha z)\left[1 - x(t-\tau)\right] - mx \\ \dot{y} & = & (1-u)\dfrac{\gamma}{2}x^2 - \beta y \\ \dot{z} & = & (1-u)\sigma x - \alpha z \end{cases} \tag{1}$$

with initial conditions $y(\theta) = y_0 \geq 0$, $z(\theta) = z_0 \geq 0$ et $x(\theta) = \varphi(\theta) \geq 0$ for $\theta \in [-\tau, 0]$ where $\varphi \in \mathcal{C}([-\tau, 0], \mathbb{R})$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^+$ equipped with the norm $||\varphi|| = \sup\limits_{-\tau \leq t \leq 0} |\varphi(t)|$.

We given the parameters values in the following table.

Our problem consists in maximizing the yield at the end of the cropping season, while controled the champignon propagation and minimizing the control cost. We propose the following objective function:

$$J(u) = \int_0^T (u(t)^2 + Bx(t))dt.$$

where B is the weight constants of control of the infected group.
The set of admissible controls is defined as follows

$$\Gamma = \{u \in L^1([0, T])/0 \leq a \leq u \leq b \leq 1; \forall t \in [0, T]\}$$

Parameter	Meaning	value	Reference
β	Sexual spores deposition rate	48 per day	F Hamelin and al, 2017 [5]
α	Asexual spores deposition rate	2	assumed
p	Sexual spores infectivity	0.01	Landry , 2015 [9]
γ	Sexual spore production rate	160 per day	Virginie and al, 2017 [12]
σ	Asexual spore production rate	20 per day	Virginie and al, 2017 [12]
m	infectious mortality rate	0.04 to 0.1	assumed

Table 1: Dimensional variables and their estimated values for M. fijiensis

The problem now is to find the control u^* satisfying $J(u^*) = \min\limits_{a \leq u \leq b} J(u)$

3. Preliminary result

In this section, we establish the existence of a solution to the delay differential system using results from R.D. Driver's text [3].
For notational purposes, we use :

$$f(t,X) = \begin{pmatrix} p(\beta y + \alpha z)(1 - x(t-\tau)) - mx \\ (1-u)\frac{\gamma}{2}x^2 - \beta y \\ (1-u)\sigma x - \alpha z \end{pmatrix}$$

where $X =^t (x, y, z)$. we can define a new function $\begin{array}{rcl} U_t : [-\tau, 0] & \longrightarrow & \mathbb{R} \\ \theta & \longrightarrow & U_t(\theta) = U(t+\theta) \end{array}$

where $U \in \{x, y, z\}$.
So the cauchy problem associed on systm (1) is given by

$$\begin{cases} \dot{X}(t) &= F(X_t) \\ X_0 &= \varphi \end{cases}$$

where $F(x_t(0), y_t(0), z_t(0), x_t(-\tau)) = f(x(t), y(t), z(t), x(t-\tau))$.
By [3], Using the fact that the right hand side of (1) has continuous partial derivatives, we obtain a local Lipschitz condition of F and we conclude for the existence of solution.

Now, we show that all these variables are always positive when time evolves.

Theorem 3.1. *For the initial conditions $(x(\theta), y(\theta), z(\theta)) \in \mathbb{R}^3_+$, the solution $(x(t), y(t), z(t))$ of system (1) are non negative and bounded for all time $t > 0$.*

Proof. see Appendix 1. □

For $u = 0$; let $R_0 = \dfrac{p\sigma}{m}$ and $S_0 = \dfrac{p(\frac{\gamma}{2} + \sigma)^2}{2\gamma m}$. Let us make the following realist assumption: $\dfrac{\gamma}{2} > \sigma$

Theorem 3.2. *1) if $R_0 < 1$ and $S_0 < 1$, the system (1) has only disease free equilibrium E_0.*

2) if $R_0 < 1$ and $S_0 > 1$, the system (1) has three equlibria point: E_0 and two endemic equilibria E_+ and E_-.

3) if $R_0 < 1$ and $S_0 = 1$, the system (1) has two equlibria point: E_0 and one endemic equilibria E^*.

4) if $R_0 \geq 1$, the system (1) has two equilibria: DFE E_0 and unique endemic equilibrium $\overline{E}$.

Proof. see Appendix 2. □

4. Existence of an optimal control

We note that the existence of an optimal control can be proved by using results from Fleming and Rishel, [(theorem 4.1 pp 68-69 [4]), for this, we will check the following properties..

1) The set of controls and corresponding state variables is non-empty.

2) The control set Γ is convex and closed.

3) The right hand side of the state system is bounded by a linear function in the state and control.

4) The integrand of the objective functional is convex on Γ and there exist non-negatif constants c_1, c_2 and $\beta > 1$ satisfying $L(u(t), X(t)) \geq c_1|u|^{\beta/2} - c_2$.

Theorem 4.1. *There exists an optimal control u^* and a corresponding solution (x^*, y^*, z^*) of the initial value problem (1) that minimizes the cost function J in Γ such that $J(u^*) = \min_{a \leq u \leq b} J(u)$.*

Proof. An existence result in Lukes ([10], Theorem 9.2.1) for the state system (1) with bounded coefficients on the finite interval time is used to give condition 1). By definition, the control set Γ is convex and closed, so condition 2) is satisfied. The right hand side of system (1) is continuous and can be written as a linear function of u with coefficients depending on time and state. Furthermore, by Γ and theorem 3.1, all variables x, y, z and u are bounded on $[0, T]$, so condition 3 hold.
For 4., the integrand function L of the objective functional is clearly convex in the controls since it is quadratic. Then, for $c_1 = 1$, and $\beta = 4$, there exist c_2 such that

$$L(t, X, u) = u^2 + Bx \geq |u|^{\beta/2} \geq c_1|u|^{\beta/2} - c_2$$

. □

5. Characterization of the optimal control

With the existence of the optimal control established, we now present the optimality system using the Pontryagin's minimum principle for delayed control problem from Göllman and al [7]. We discuss the theorem that relates to the characterization of the optimal control. The optimality system can be used to compute candidates for optimal control. To do this, we begin by defining a augmented Hamiltonian.

For the adjoint variables $\Lambda = (\lambda_1, \lambda_2, \lambda_3)$ and penalty multipliers, $w_i(t)$, i=1,2 ; that are attached to the control constraints; We define the augmented Hamiltonian as follows:

$$\begin{aligned} H(u(t), X(t), x(t-\tau), \Lambda(t)) &= \left[u(t)^2 + Bx(t)\right] + \lambda_1(t)\left[p(\beta y(t) + \alpha z(t))(1 - x(t-\tau)) - mx(t)\right] \\ &\quad + \lambda_2(t)\left[(1-u(t))\frac{\gamma}{2}x(t)^2 - \beta y(t)\right] + \lambda_3(t)\left[(1-u(t))\sigma x(t) - \alpha z(t)\right] \\ &\quad + w_1(a - u(t)) + w_2(u(t) - b), \end{aligned}$$

where w_1 et w_2 are non-negatif and satifiyng $w_1(a - u^*(t)) = w_2(u^*(t) - b) = 0$ for optimal control u^*.

Theorem 5.1. *Given optimal control u^* and the corresponding solutions x^*, y^*, z^* of system (1), there exist adjoint variables λ_1, λ_2 and λ_3 verifiyng the system:*

$$\left\{ \begin{array}{lcl} \dot{\lambda_1}(t) &=& -B + \lambda_1 m - \lambda_2(t)(1-u^*)\frac{\gamma}{2}x^*(t) - \lambda_3(t)(1-u^*)\sigma \\ && + \mathcal{X}_{[t_0, T-\tau]}(t)\lambda_1(t+\tau)p(\beta y^*(t+\tau) + \alpha z^*(t+\tau)) \\ \dot{\lambda_2}(t) &=& -\lambda_1(t)\beta p(1 - x^*(t-\tau)) + \beta\lambda_2(t) \\ \dot{\lambda_3}(t) &=& -\lambda_1(t)\alpha p(1 - x^*(t-\tau)) + \alpha\lambda_3(t) \end{array} \right. \quad (2)$$

with transversality conditions $\lambda_i(T) = 0; i = 1, 2, 3.$

Furthermore, u^* are represented by $\quad u^* = \max\{a, \min\{b, \frac{1}{4}x^*(\gamma x^*\lambda_2 + 2\sigma\lambda_3)\}\}$

Proof. Using the Pontryagin's minimum principle for delayed control problem from Göllman and al [7], adjoint differential equation is giving by

$$\left\{ \begin{array}{lcl} \dot{\lambda_1}(t) &=& -\frac{\partial H}{\partial x(t)}(t) - \mathcal{X}_{[t_0, T-\tau]}(t)\frac{\partial H}{\partial x(t-\tau)}(u(t+\tau), X(t+\tau), x(t), \Lambda(t+\tau)) \\ \dot{\lambda_2}(t) &=& -\frac{\partial H}{\partial y(t)}(u(t), X(t), x(t-\tau), \Lambda(t)) \\ \dot{\lambda_3}(t) &=& -\frac{\partial H}{\partial z(t)}(u(t), X(t), x(t-\tau), \Lambda(t)) \end{array} \right.$$

with traversality conditions $\lambda_i(T) = 0; i = 1, 2, 3.$ where $\mathcal{X}_{[t_0, T-\tau]}$ is the "indicatrice" function on $[t_0, T-\tau]$, hence we obtain system(2).

Also, the optimal control u^* can be solved from the optimality condition $\frac{\partial H}{\partial u} = 0$, that give $u^* = \frac{1}{4}x^*(\gamma x^*\lambda_2 + 2\sigma\lambda_3) + w_1 - w_2$.

To determine an explicit expression for the optimal control without w_1 and w_2 a standard optimality technique is utilized. We consider the following three cases.

i) On the set $\{t : a < u^*(t) < b\}$, we have $w_1 = w_2 = 0$ and $u^*(t) = \frac{1}{4}x^*(t)(\gamma x^*(t)\lambda_2(t) + 2\sigma\lambda_3(t))$.

ii) On the set $\{t : u^*(t) = a\}$, we have $w_2(t) = 0$. Hence $a = u^* = \frac{1}{4}x^*(\gamma x^*\lambda_2 + 2\sigma\lambda_3) + w_1$. Therefore $\frac{1}{4}x^*(\gamma x^*\lambda_2 + 2\sigma\lambda_3) \leq a$ since $w_1(t) \geq 0$.

iii) On the set $\{t : u^*(t) = b\}$, we have $w_1(t) = 0$. Hence $b = u^* = \frac{1}{4}x^*(\gamma x^*\lambda_2 + 2\sigma\lambda_3) - w_2$. Therefore $\frac{1}{4}x^*(\gamma x^*\lambda_2 + 2\sigma\lambda_3) \geq b$ because $w_2 \geq 0$.

Thus i), ii) and iii) can resume at

$$u^*(t) = \max\{a, \min\{b, \frac{1}{4}x^*(t)(\gamma x^*(t)\lambda_2(t) + 2\sigma\lambda_3(t))\}\}$$

□

6. Numerical simulations

In this section, we present the numerical solution of our control problem and compare it with the solution in the absence of control. We adopted the forward-backward sweep method to solve numerically our optimal model with time delay [8]. After making an initial guess for the control functions, we first solved the initial valued state system forward in time. Then, using the same initial guess for the control functions, we solve the adjoint system with terminal conditions backward in time. The controls are updated in each iteration using the optimality conditions.

The parameters used in solving the optimality system are estimates of data from BLSD as summarized in Table 1 with $m = 0.08$, $\tau = 15$, $a = 0$ and $b = 1$. With this values, the threshold give $R_0 = 2.5$. The cropping season of banana variyng to 10 at 12 months [13], so we take $T = 365$ days.

The simulations plots are given in figure1 for $B = 2$ (A, B, C, D) and figure(2) for $B = 5$ (a, b, c, d) where We compare the cases with control (red curves) and without control (blue curves). We observe that the control $u(t)$ (see (D) and d)) reduces the disease but its effectiveness depends of parameter B which can represent the fungicides concentration on the leaf in the case of chemical control.

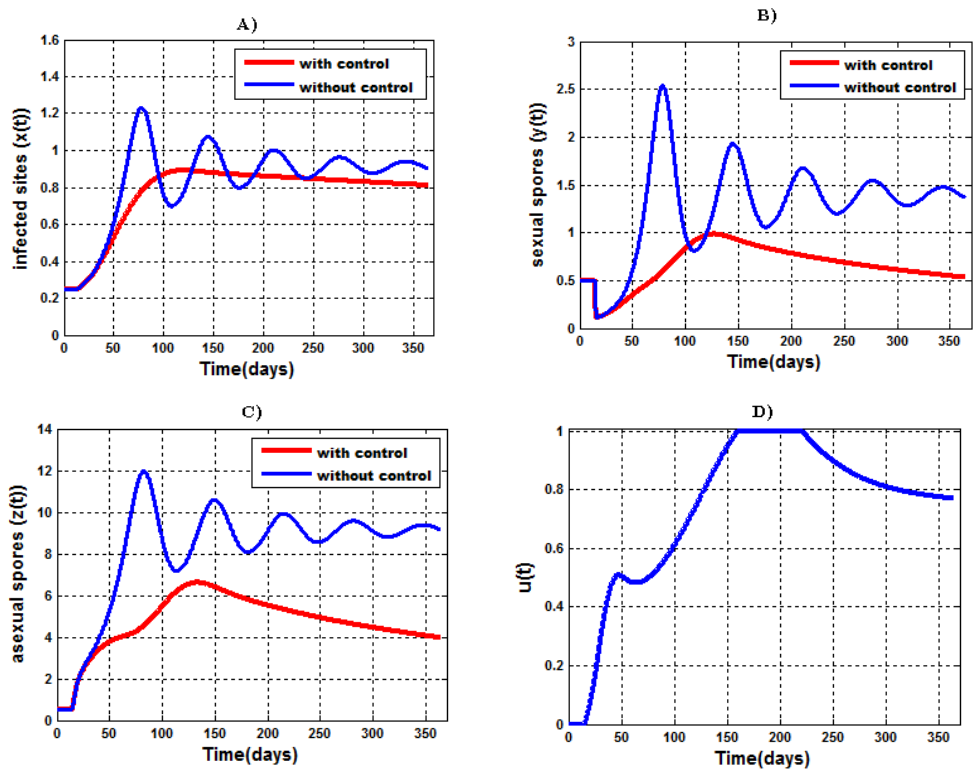

Figure 1: Simulation for $B = 2$. We remark that the control is less effective because the weight of control B on the infected leaves is law, nevertheless we can see that the control $u(t)$ (see D)) grows weakly up to 160 days where it reaches its maximum and remains there up to 225 days (which corresponds to the flowering time[13]) and begins to decrease.

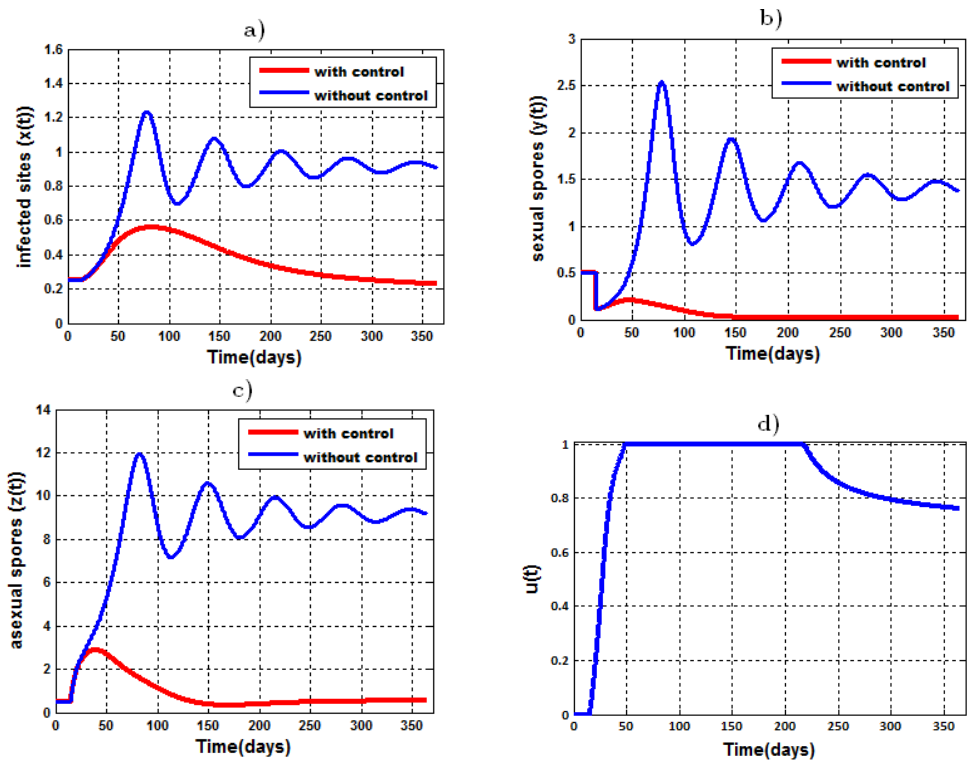

Figure 2: Simulation of system (1) with $B = 5$. We can remark that the control is more effective because the weight of infected leaves is high. We can see that the control $u(t)$ (see d)) grows rapidly and reaches its maximum from 50 days and remains so until 230 days(which corresponds to the flowering time [13]) before starting to decrease slowly.

7. Conclusion

In this work, we have proposed a mathematical pathogen-host model with time delay of banana black leaf streak disease which takes into account control strategies to reduce the spores production. We have designed an optimal control problem that consists to maximizing the yield at the end of the cropping season, while controled the champignon propagation and minimizing the control cost we have defined. We have characterized the optimal control using the Pontryagin's maximum principle and we have solved numerically our system. For this numerical simulations, we saw that the control reduces the disease and its effectiveness depends of parameter B of our cost function which can represent the constant weight of control on the infected leaf.

8. References

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Appendix 1.Proof of theorem(3.1)

The model system can be put into the matrix form $X = g(X)$ where $X = (x, y, z)^T$ and

$$g(X) = \begin{pmatrix} g_1(X) \\ g_2(X) \\ g_3(X) \end{pmatrix} = \begin{pmatrix} (p\beta y + q\alpha z)(1 - x(t-\tau)) - mx \\ (1-u)\frac{\gamma}{2}x^2 - \beta y \\ (1-u)\sigma x - \alpha z \end{pmatrix}.$$

We have $g_1(0, z, y) \geq 0$, $g_2(x, 0, z) \geq 0$ and $g_3(x, y, 0) \geq 0$. Hence, every solution which starts in $\mathbb{R}^3_+$ will remain in $\mathbb{R}^3_+$.
We prove now that the solutions of model system(1) are bounded. using the fact that x is continous over the finite interval $[0, T]$, we conclude that x has a maximum x^* in $[0, T]$.

By equation(2) of system(1), we have $\dot{y}(t) \leq \frac{\gamma x^*}{2} - \beta y(t)$ which implies that $y(t) \leq \frac{x^* \gamma}{2\beta}$ and the same by eqyation(3) of system(1), we deduce that $z(t) \leq \frac{\sigma x^*}{\alpha}$, therefore x, y and z are bounded.

Appendix 2.Proof of theorem(3.2)

For $u = 0$ and $p = q$, the equilibria of system(1) are the solutions of system:

$$\begin{cases} p(\beta y + \alpha z)(1-x) - mx = 0 & (1) \\ \frac{\gamma}{2}x^2 - \beta y = 0 & (2) \\ \sigma x - \alpha z = 0 & (3) \end{cases}$$

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From (2) and (3), whe have $y = \frac{\gamma}{2}x^2$ and $z = \frac{\sigma}{\alpha}x$. Introduce in (1), we obtain

$$x = 0 \quad or \quad \frac{p\gamma}{2}x^2 - p(\frac{\gamma}{2} - \sigma)x + m - p\sigma = 0. \quad (E)$$

For $x = 0$, we have $y = z = 0$ and whe obtain disease free equilibrium $E_0 = (0, 0, 0)$. If $m - p\sigma \leq 0$ or $R_0 = \frac{p\sigma}{m} \geq 1$, then using the Descarte sign ruler's, equation (E) has a unique positive solution $\overline{x}$; then $\overline{y} = \frac{\gamma}{2\beta}\overline{x}^2$, and $\overline{z} = \frac{\sigma}{\alpha}\overline{x}$, hence endemic equilibrium given for $R_0 \geq 1$.

If $R_0 < 1$, then the discriminant of equation (E) is given by $\Delta = p^2(\frac{\gamma}{2} + \sigma)^2 - 2p\gamma m$. Hence, if $S_0 > 1$, i.e. $\Delta > 0$ then (E) has two positive solutions x_+ and x_- and we obtain equilibria E_+ and E_- with $y_\pm = \frac{\gamma}{2}x_\pm^2$ and $z_\pm = \frac{\sigma}{\alpha}x_\pm$. Finally if $S_0 = 1$ that implies $\Delta = 0$, (E) has a unique positive solution x^* and we have equilibrium E^* with $y^* = \frac{\gamma}{2}x^{*2}$ and $z^* = \frac{\sigma}{\alpha}x^*$.